Clifford Valued Differential Forms, and Some Issues in Gravitation, Electromagnetism and "Unified" Theories*

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Abstract

In this paper we show how to describe the general theory of a linear metric compatible connection with the theory of Clifford valued differential forms. This is done by realizing that for each spacetime point the Lie algebra of Clifford bivectors is isomorphic to the Lie algebra of $Sl(2,\mathbb{C})$. In that way the pullback of the linear connection under a local trivialization of the bundle (i.e., a choice of gauge) is represented by a Clifford valued 1-form. That observation makes it possible to realize immediately that Einstein's gravitational theory can be formulated in a way which is similar to a $Sl(2,\mathbb{C})$ gauge theory. Such a theory is compared with other interesting mathematical formulations of Einstein's theory. and particularly with a supposedly "unified" field theory of gravitation and electromagnetism proposed by M. Sachs. We show that his identification of Maxwell equations within his formalism is not a valid one. Also, taking profit of the mathematical methods introduced in the paper we investigate a very polemical issue in Einstein gravitational theory, namely the problem of the 'energy-momentum' conservation. We show that many statements appearing in the literature are confusing or even wrong.

1 Introduction

In this paper we introduce the concept of Clifford valued differential forms[1], mathematical entities which are sections of $\mathcal{C}\ell(TM)\otimes\bigwedge T^*M$. We show how

[1]Analogous, but non equivalent concepts have been introduced in [13, 64, 66, 63]. In particular [13] introduce clifforms, i.e., forms with values in a abstract (internal) Clifford algebra $\mathbb{R}_{p,q}$ associated with a pair $(\mathbb{R}^n, g)$, where $n = p + q$ and g is a bilinear form of signature (p, q) in $\mathbb{R}^n$. These objects *differ* from the Clifford valued differential forms used in this text., whith dispenses any abstract (internal) space.

with the aid of this concept we can produce a very beautiful *description* of the theory of linear connections, where the representative of a given linear connection in a given *gauge* is given by a *bivector* valued 1-form. For that objects we introduce the concept of *exterior* covariant differential and *extended* covariant derivative operators. Our *natural* definitions[2] are to be compared with other approaches on related subjects (as described, e.g., in [2, 3, 26, 28, 44, 46, 59]) and have been designed in order to parallel in a noticeable way the formalism of the theory of connections in principal bundles and their associated covariant derivative operators acting on associated vector bundles. We identify Cartan curvature 2-forms and *curvature bivectors*. The curvature 2-forms satisfy Cartan's second structure equation and the curvature bivectors satisfy equations in complete analogy with equations of gauge theories. This immediately suggests to write Einstein's theory in that formalism, something that has already been done and extensively studied in the past (see e.g., [6, 8]). Our methodology however, suggests new ways of taking advantage of such a formulation, but this is postponed for a later paper. Here, our investigation of the $Sl(2,\mathbb{C})$ nonhomogeneous gauge equation for the curvature bivector is restricted to the relationship between that equation and some other equations which use different formalisms, but which express the same information as the one contained in the original Einstein's equations written in classical tensor formalism. Our analysis includes a careful investigation of the relationship of the $Sl(2,\mathbb{C})$ nonhomogeneous gauge equation for the curvature bivector and some interesting equations appearing in M. Sachs theory [51, 52, 53].

We already showed in [45] that unfortunately M. Sachs identified equivocally his basic variables q_μ as being quaternion fields over a Lorentzian spacetime. Well, they are *not*. The real mathematical structure of these objects is that they are matrix representations of particular sections of the even Clifford bundle of multivectors $\mathcal{C}\ell(TM)$ (called paravector fields in mathematical literature). Here we show that the identification proposed M. Sachs of a new antisymmetric field [51, 52, 53] in his 'unified' theory as an electromagnetic field is, according to our opinion, an equivocated one. Indeed, as will be proved in detail, M. Sachs 'electromagnetic fields' $\mathbb{F}_{\mathbf{ab}}$ (whose precise mathematical nature is disclosed below) are nothing more than some combinations of the curvature bivectors[3], objects that appear naturally when we try to formulate Einstein's gravitational theory as a $Sl(2,\mathbb{C})$ gauge theory. The equations found by M. Sachs, which are satisfied by his antisymmetric fields $\mathbb{F}_{\mathbf{ab}}$ looks like Maxwell equations in components, but they are *not* Maxwell equations. However, we can say that they do reveal one more of the many faces of Einstein's equations[4].

Taking profit of the mathematical methods introduced in this paper we also discuss some controversial, but conceptually important issues concerning the law of energy-momentum conservation in General Relativity, showing that many

[2]Our defintions, for the best of our knowledge, appears here for the first time.

[3]The curvature bivectors are physically and mathematically equivalent to the Cartan curvature 2-forms, since they carry the same information. This statement will become obvious from our study in section 3.4.

[4]Some other faces of that equations are shown in the Appendices.

statements appearing in the literature are confusing and even wrong.

The paper contains two Appendices. Appendix A recalls some results of the Clifford calculus necessary for the calculations presented in the main text. Appendix B recalls the correct intrinsic presentation of Einstein's equations in terms of tetrad fields $\{\theta^{\mathbf{a}}\}$, when these fields are sections of the Clifford bundle and compare that equations, which some other equations for that objects recently presented in the literature.

2 Recall of Some Facts of the Theory of Linear Connections

2.1 Preliminaries

In the general theory of connections [9, 31] a connection is a 1-form in the cotangent space of a principal bundle, with values in the Lie algebra of a gauge group. In order to develop a theory of a linear connection[5]

$$\overset{\blacktriangle}{\boldsymbol{\omega}} \in \sec T^{*}P_{\mathrm{SO}_{1,3}^{e}}(M) \otimes \mathrm{sl}(2,\mathbb{C}), \tag{1}$$

with an *exterior* covariant derivative operator acting on sections of associated vector bundles to the principal bundle $P_{\mathrm{SO}_{1,3}^{e}}(M)$ which reproduces moreover the well known results obtained with the usual covariant derivative of tensor fields in the base manifold, we need to introduce the concept of a *soldering* form

$$\overset{\blacktriangle}{\boldsymbol{\theta}} \in \sec T^{*}P_{\mathrm{SO}_{1,3}^{e}}(M) \otimes \mathbb{R}^{1,3}. \tag{2}$$

Let be $U \subset M$ and let $\varsigma : U \rightarrow \varsigma(U) \subset P_{\mathrm{SO}_{1,3}^{e}}(M)$.We are interested in the pullbacks $\varsigma^{*}\overset{\blacktriangle}{\boldsymbol{\omega}}$ and $\varsigma^{*}\overset{\blacktriangle}{\boldsymbol{\theta}}$ once we give a local trivialization of the respective bundles. As it is well known [9, 31], in a local chart $\langle x^{\mu}\rangle$ covering U, $\varsigma^{*}\ \overset{\blacktriangle}{\boldsymbol{\theta}}$ uniquely determines

$$\boldsymbol{\theta} = e_{\mu} \otimes dx^{\mu} \equiv e_{\mu}dx^{\mu} \in \sec TM \otimes \bigwedge\nolimits^{1} T^{*}M. \tag{3}$$

Now, we give the Clifford algebra structure to the *tangent bundle*, thus generating the Clifford bundle $\mathcal{C}\ell(TM) = \bigcup_{x} \mathcal{C}\ell_{x}(M)$, with $\mathcal{C}\ell_{x}(M) \simeq \mathbb{R}_{1,3}$ introduced in Appendix A.

We recall moreover, a well known result [34], namely, that for each $x \in U \subset M$ the bivectors of $\mathcal{C}\ell(T_{x}M)$ generate under the product defined by the commutator, the Lie algebra $\mathrm{sl}(2,\mathbb{C})$. We thus are lead to define the representatives in $\mathcal{C}\ell(TM) \otimes \bigwedge T^{*}M$ for $\boldsymbol{\theta}$ and for the the pullback $\boldsymbol{\omega}$ of the connection *in a*

[5] In words, $\overset{\blacktriangle}{\boldsymbol{\omega}}$ is a 1-form in the cotangent space of the bundle of ortonornal frames with values in the Lie algebra $\mathrm{so}_{1,3}^{e} \simeq \mathrm{sl}(2,\mathbb{C})$ of the group $\mathrm{SO}_{1,3}^{e}$.

given gauge (that we represent with the same symbols):

$$\begin{aligned}
\boldsymbol{\theta} &= e_\mu dx^\mu = \mathbf{e_a}\theta^{\mathbf{a}} \in \sec \bigwedge\nolimits^1 TM \otimes \bigwedge\nolimits^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge\nolimits^1 T^*M, \\
\boldsymbol{\omega} &= \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}}\mathbf{e_b}\mathbf{e_c}\theta^{\mathbf{a}} \\
&= \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}}(\mathbf{e_b} \wedge \mathbf{e_c}) \otimes \theta^{\mathbf{a}} \in \sec \bigwedge\nolimits^2 TM \otimes \bigwedge\nolimits^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge\nolimits^1 T^*M. \quad (4)
\end{aligned}$$

Before we continue we must recall that whereas $\boldsymbol{\theta}$ is a true tensor, $\boldsymbol{\omega}$ is not a true tensor, since as it is well known, its 'components' do not have the tensor transformation properties. Note that the $\omega_{\mathbf{a}}^{\mathbf{bc}}$ are the 'components' of the connection defined by

$$D_{\mathbf{e_a}}\mathbf{e^b} = -\omega_{\mathbf{ac}}^{\mathbf{b}}\mathbf{e^c}, \quad \omega_{\mathbf{abc}} = -\omega_{\mathbf{cba}} = \eta_{\mathbf{ad}}\omega_{\mathbf{bc}}^{\mathbf{d}}, \tag{5}$$

where $D_{\mathbf{e_a}}$ is a metric compatible covariant derivative operator[6] defined on the tensor bundle, that naturally acts on $\mathcal{C}\ell(TM)$ (see, e.g., [11]). Objects like $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ will be called Clifford valued differential forms (or Clifford valued forms, for short), and in sections 3 and 4 we give a detailed account of the algebra and calculus of that objects. But, before we start this project we need to recall some concepts of the theory of linear connections.

2.2 Exterior Covariant Differential

One of our objectives is to show how to describe, with our formalism an exterior covariant differential (*EXCD*) which acts naturally on sections of Clifford valued differential forms (i.e., sections of $\sec \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$) and which *mimics* the action of the pullback of the exterior covariant derivative operator acting on sections of a vector bundle associated to the principal bundle $P_{\mathrm{SO}^e_{1,3}}(M)$, once a linear metric compatible connection is given. Our motivation for the definition of the *EXCD* is that with it, the calculations of curvature bivectors, Bianchi identities, etc., use always the same formula. Of course, we compare our definition, with other definitions of analogous, but distinct concepts, already used in the literature, showing where they differ from ours, and why we think that ours seems more appropriate. In particular, with the *EXCD* and its associated *extended* covariant derivative (*ECD*) we can write Einstein's equations in such a way that the resulting equation looks like an equation for a gauge theory of the group $Sl(2,\mathbb{C})$. To achieve our goal, we recall below the well known definition of the exterior covariant differential $\mathbf{d}^E$ acting on arbitrary sections of a vector bundle $E(M)$ (associated to $P_{\mathrm{SO}^e_{1,3}}(M)$ and having as typical fiber a l-dimensional real vector space) and on $\mathrm{end}E\,(M) = E\,(M) \otimes E^*(M)$, the bundle of endomorphisms of $E\,(M)$. We recall also the concept of absolute differential acting on sections of the tensor bundle, for the particular case of $\bigwedge^l TM$.

[6] After section 3.4, $D_{\mathbf{e_a}}$ refers to the Levi-Civita covariant derivative operator.

Definition 1 *The exterior covariant differential operator (ECDO)* $\mathbf{d}^E$ *acting on sections of* $E(M)$ *and* $\mathrm{end}E(M)$ *is the mapping*

$$\mathbf{d}^E\colon \sec E(M) \to \sec E(M) \otimes \bigwedge^1 T^*M, \tag{6}$$

such that for any differentiable function $f : M \to \mathbb{R}$, $A \in \sec E(M)$ *and any* $F \in \sec(\mathrm{end}E(M) \otimes \bigwedge^p T^*M)$, $G \in \sec(\mathrm{end}E(M) \otimes \bigwedge^q T^*M)$ *we have:*

$$\begin{aligned}
\mathbf{d}^E(fA) &= df \otimes A + f\mathbf{d}^E A, \\
\mathbf{d}^E(F \otimes_\wedge A) &= \mathbf{d}^E F \otimes_\wedge A + (-1)^p F \otimes_\wedge \mathbf{d}^E A, \\
\mathbf{d}^E(F \otimes_\wedge G) &= \mathbf{d}^E F \otimes_\wedge G + (-1)^p F \otimes_\wedge \mathbf{d}^E G.
\end{aligned} \tag{7}$$

In Eq.(7), writing $F = F^a \otimes f_a^{(p)}$, $G = G^b \otimes g_b^{(q)}$ where F^a, $G^b \in \sec(\mathrm{end}E(M))$, $f_a^{(p)} \in \sec \bigwedge^p T^*M$ and $g_b^{(q)} \in \sec \bigwedge^q T^*M$ we have

$$\begin{aligned}
F \otimes_\wedge A &= \left(F^a \otimes f_a^{(p)}\right) \otimes_\wedge A, \\
F \otimes_\wedge G &= \left(F^a \otimes f_a^{(p)}\right) \otimes_\wedge G^b \otimes g_b^{(q)}.
\end{aligned} \tag{8}$$

In what follows, in order to simplify the notation we eventually use when there is no possibility of confusion, the simplified (sloppy) notation

$$\begin{aligned}
(F^a A) \otimes f_a^{(p)} &\equiv (F^a A)\, f_a^{(p)}, \\
\left(F^a \otimes f_a^{(p)}\right) \otimes_\wedge G^b \otimes g_b^{(q)} &= \left(F^a G^b\right) f_a^{(p)} \wedge g_b^{(q)},
\end{aligned} \tag{9}$$

where $F^a A \in \sec E(M)$ and $F^a G^b$ means the composition of the respective endomorphisms.

Let $U \subset M$ be an open subset of M, $\langle x^\mu \rangle$ a coordinate functions of a maximal atlas of M, $\{e_\mu\}$ a coordinate basis of $TU \subset TM$ and $\{s_\mathbf{K}\}$, $\mathbf{K} = 1, 2, ...l$ a basis for any $\sec E(U) \subset \sec E(M)$. Then, a basis for any section of $E(M) \otimes \bigwedge^1 T^*M$ is given by $\{s_\mathbf{K} \otimes dx^\mu\}$.

Definition 2 *The covariant derivative operator* $D_{e_\mu} : \sec E(M) \to \sec E(M)$ *is given by*

$$\mathbf{d}^E A \doteq \left(D_{e_\mu} A\right) \otimes dx^\mu, \tag{10}$$

where, writing $A = A^\mathbf{K} \otimes s_\mathbf{K}$ *we have*

$$D_{e_\mu} A = \partial_\mu A^\mathbf{K} \otimes s_\mathbf{K} + A^\mathbf{K} \otimes D_{e_\mu} s_\mathbf{K}. \tag{11}$$

Now, let examine the case where $E(M) = TM \equiv \bigwedge^1 (TM) \hookrightarrow \mathcal{C}\ell(TM)$. Let $\{\mathbf{e_j}\}$, be an orthonormal basis of TM. Then,using Eq.(11) and Eq.(5)

$$\begin{aligned}
\mathbf{d}^E \mathbf{e_j} &= (D_{e_\mathbf{k}} \mathbf{e_j}) \otimes \theta^\mathbf{k} \equiv \mathbf{e_k} \otimes \boldsymbol{\omega}_\mathbf{j}^\mathbf{k} \\
\boldsymbol{\omega}_\mathbf{j}^\mathbf{k} &= \boldsymbol{\omega}_\mathbf{rj}^\mathbf{k} \theta^\mathbf{r},
\end{aligned} \tag{12}$$

where the $\boldsymbol{\omega}_{\mathbf{j}}^{\mathbf{k}} \in \sec \bigwedge^1 T^*M$ are the so-called *connection 1-forms*.

Also, for $\mathbf{v} = v^{\mathbf{i}}\mathbf{e}_{\mathbf{i}} \in \sec TM$, we have

$$\begin{aligned} \mathbf{d}^E\mathbf{v} &= D_{\mathbf{e}_{\mathbf{i}}}\mathbf{v}\otimes\theta^{\mathbf{i}} = \mathbf{e}_{\mathbf{i}} \otimes \mathbf{d}^E v^{\mathbf{i}}, \\ \mathbf{d}^E v^{\mathbf{i}} &= dv^{\mathbf{i}} + \boldsymbol{\omega}_{\mathbf{k}}^{\mathbf{i}} v^{\mathbf{k}}. \end{aligned} \tag{13}$$

2.3 Absolute Differential

Now, let $E(M) = TM \equiv \bigwedge^l (TM) \hookrightarrow \mathcal{C}\ell(TM)$. Recall that the usual *absolute differential* D of $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ is a mapping (see, e.g., [9])

$$D\colon \sec \bigwedge^l TM \to \sec \bigwedge^l TM \otimes \bigwedge^1 T^*M, \tag{14}$$

such that for any differentiable $A \in \sec \bigwedge^l TM$ we have

$$DA = (D_{\mathbf{e}_{\mathbf{i}}}A) \otimes \theta^i, \tag{15}$$

where $D_{e_{\mathbf{i}}}A$ is the standard covariant derivative of $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$. Also, for any differentiable function $f : M \to \mathbb{R}$, and differentiable $A \in \sec \bigwedge^l TM$ we have

$$D(fA) = df \otimes A + fDA. \tag{16}$$

Now, if we suppose that the orthonormal basis $\{\mathbf{e}_{\mathbf{j}}\}$ of TM is such that each $\mathbf{e}_{\mathbf{j}} \in \sec \bigwedge^1 TM \hookrightarrow \sec \mathcal{C}\ell(TM)$, we can find easily using the Clifford algebra structure of the space of multivectors that Eq.(12) can be written as:

$$\begin{aligned} D\mathbf{e}_{\mathbf{j}} &= (D_{\mathbf{e}_{\mathbf{k}}}\mathbf{e}_{\mathbf{j}})\theta^{\mathbf{k}} = \frac{1}{2}[\boldsymbol{\omega}, \mathbf{e}_{\mathbf{j}}] = -\mathbf{e}_{\mathbf{j}} \lrcorner \boldsymbol{\omega} \\ \boldsymbol{\omega} &= \frac{1}{2}\omega_{\mathbf{k}}^{\mathbf{ab}} \mathbf{e}_{\mathbf{a}} \wedge \mathbf{e}_{\mathbf{b}} \otimes \theta^{\mathbf{k}} \\ &\equiv \frac{1}{2}\omega_{\mathbf{k}}^{\mathbf{ab}} \mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} \otimes \theta^{\mathbf{k}} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M, \end{aligned} \tag{17}$$

where $\boldsymbol{\omega}$ is the *representative* of the connection in a given gauge.

The general case is given by the following proposition.

Proposition 3 *For $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ we have*

$$DA = dA + \frac{1}{2}[\boldsymbol{\omega}, A]. \tag{18}$$

Proof. The proof is a simple calculation, left to the reader. ■

Eq.(18) can now be extended by linearity for an arbitrary nonhomogeneous multivector $A \in \sec \mathcal{C}\ell\,(TM)$.

Remark 4 *We see that when* $E(M) = \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell\,(TM)$ *the absolute differential* D *can be identified with the exterior covariant derivative* $\mathbf{d}^E$.

We proceed now to find an appropriate *exterior* covariant differential which acts naturally on Clifford valued differential forms, i.e., objects that are sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$ $(\equiv \bigwedge T^*M \;\; \otimes \mathcal{C}\ell(TM))$ (see next section). Note that we cannot simply use the above definition by using $E\,(M) = \mathcal{C}\ell(TM)$ and end$E\,(M) =$ end$\mathcal{C}\ell(TM)$, because end$\mathcal{C}\ell(TM) \neq \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$. Instead, we must use the above theory and possible applications as a guide in order to find an appropriate definition. Let us see how this can be done.

3 Clifford Valued Differential Forms

Definition 5 *A homogeneous multivector valued differential form of type* (l, p) *is a section of* $\bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$, *for* $0 \leq l \leq 4$, $0 \leq p \leq 4$. *A section of* $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$ *such that the multivector part is non homogeneous is called a Clifford valued differential form.*

We recall, that any $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ can always be written as

$$\begin{aligned} A &= m_{(l)} \otimes \psi^{(p)} \equiv \frac{1}{l!} m_{(l)}^{\mathbf{i}_1 \ldots \mathbf{i}_l} \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_l} \otimes \psi^{(p)} \\ &= \frac{1}{p!} m_{(l)} \otimes \psi^{(p)}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \theta^{\mathbf{j}_1} \wedge \ldots \wedge \theta^{\mathbf{j}_p} \\ &= \frac{1}{l!p!} m_{(l)}^{\mathbf{i}_1 \ldots \mathbf{i}_l} \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_l} \otimes \psi^{(p)}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \theta^{\mathbf{j}_1} \wedge \ldots \wedge \theta^{\mathbf{j}_p} \\ &= \frac{1}{l!p!} A^{\mathbf{i}_1 \ldots \mathbf{i}_l}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_l} \otimes \theta^{\mathbf{i}_1} \wedge \ldots \wedge \theta^{\mathbf{i}_p}. \end{aligned} \tag{19}$$

Definition 6 *The* $\otimes_\wedge$ *product of* $A = \overset{m}{A} \otimes \psi^{(p)} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ *and*

$B = \overset{m}{B} \otimes \chi^{(p)} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$ *is the mapping*[7]*:*

$$\begin{aligned} \otimes_\wedge &: \sec \mathcal{C}\ell(TM) \otimes \bigwedge^l T^*M \times \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M \\ &\rightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{l+p} T^*M, \\ A \otimes_\wedge B &= \overset{m}{A}\overset{m}{B} \otimes \psi^{(p)} \wedge \chi^{(q)}. \end{aligned} \tag{20}$$

Definition 7 *The commutator* $[A, B]$ *of* $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ *and* $B \in \bigwedge^m TM \otimes \bigwedge^q T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$ *is the mapping:*

$$\begin{aligned} [\quad,\quad] &: \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \times \sec \bigwedge^m TM \otimes \bigwedge^q T^*M \\ &\rightarrow \sec((\sum_{k=|l-m|}^{|l+m|} \bigwedge^k T^*M) \otimes \bigwedge^{p+q} T^*M) \\ [A, B] &= A \otimes_\wedge B - (-1)^{pq}\, B \otimes_\wedge A \end{aligned} \tag{21}$$

Writing $A = \frac{1}{l!} A^{\mathbf{j}_1 \ldots \mathbf{j}_l} \mathbf{e}_{\mathbf{j}_1} \ldots \mathbf{e}_{\mathbf{j}_l} \psi^{(p)}$*,* $B = \frac{1}{m!} B^{\mathbf{i}_1 \ldots \mathbf{i}_m} \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_m} \chi^{(q)}$*, with* $\psi^{(p)} \in \sec \bigwedge^p T^*M$ *and* $\chi^{(q)} \in \sec \bigwedge^q T^*M$*, we have*

$$[A, B] = \frac{1}{l!m!} A^{\mathbf{j}_1 \ldots \mathbf{j}_l} B^{\mathbf{i}_1 \ldots \mathbf{i}_m} \left[\mathbf{e}_{\mathbf{j}_1} \ldots \mathbf{e}_{\mathbf{j}_l}, \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_m}\right] \psi^{(p)} \wedge \chi^{(q)}. \tag{22}$$

The definition of the commutator is extended by linearity to arbitrary sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$*.*

Now, we have the proposition.

Proposition 8 *Let* $A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$*,* $B \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$*,* $C \in A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^r T^*M$*. Then,*

$$[A, B] = (-1)^{1+pq}[B, A], \tag{23}$$

and

$$(-1)^{pr}\left[[A, B], C\right] + (-1)^{qp}\left[[B, C], A\right] + (-)^{rq}\left[[C, A], B\right] = 0. \tag{24}$$

Proof. It follows directly from a simple calculation, left to the reader. ■

Eq.(24) may be called the *graded Jacobi identity* [4].

[7] $\overset{m}{A}$ and $\overset{m}{B}$ are general nonhomogeous multivector fields.

Corollary 9 *Let be* $A^{(2)} \in \sec \bigwedge^2 (TM) \otimes \bigwedge^p T^*M$ *and* $B \in \sec \bigwedge^r (TM) \otimes \bigwedge^q T^*M$. *Then,*

$$[A^{(2)}, B] = C, \tag{25}$$

where $C \in \sec \bigwedge^r (TM) \otimes \bigwedge^{p+q} T^*M$.

Proof. It follows from a direct calculation, left to the reader.■

Proposition 10 *Let* $\omega \in \sec \bigwedge^2 (TM) \otimes \bigwedge^1 T^*M$, $A \in \sec \bigwedge^l (TM) \otimes \bigwedge^p T^*M . B \in \sec \bigwedge^m (TM) \otimes \bigwedge^q T^*M$. *Then, we have*

$$[\omega, A \otimes_\wedge B] = [\omega, A] \otimes_\wedge B + (-1)^p A \otimes_\wedge [\omega, B]. \tag{26}$$

Proof. Using the definition of the commutator we can write

$$\begin{aligned}
[\boldsymbol{\omega}, A] \otimes_\wedge B &= (\boldsymbol{\omega} \otimes_\wedge A - (-1)^p A \otimes_\wedge \boldsymbol{\omega}) \otimes_\wedge B \\
&= (\boldsymbol{\omega} \otimes_\wedge A \otimes_\wedge B - (-1)^{p+q} A \otimes_\wedge B \otimes_\wedge \boldsymbol{\omega}) \\
&+ (-1)^{p+q} A \otimes_\wedge B \otimes_\wedge \boldsymbol{\omega} - (-1)^p A \otimes_\wedge \boldsymbol{\omega} \otimes_\wedge B \\
&= [\boldsymbol{\omega}, A \otimes_\wedge B] - (-1)^p A \otimes_\wedge [\boldsymbol{\omega}, B],
\end{aligned}$$

from where the desired result follows. ■

From Eq.(**??**) we have also[8]

$$\begin{aligned}
&(p+q)[\boldsymbol{\omega}, A \otimes_\wedge B] \\
&= p[\boldsymbol{\omega}, A] \otimes_\wedge B + (-1)^p q A \otimes_\wedge [\boldsymbol{\omega}, B] \\
&+ q[\boldsymbol{\omega}, A] \otimes_\wedge B + (-1)^p p A \otimes_\wedge [\boldsymbol{\omega}, B].
\end{aligned}$$

Definition 11 *The action of the differential operator d acting on*

$$A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M,$$

is given by:

$$\begin{aligned}
dA &\doteq \mathbf{e}_{\mathbf{j}_1} ... \mathbf{e}_{\mathbf{j}_l} \otimes dA^{\mathbf{j}_1 \cdots \mathbf{j}_l} \\
&= \mathbf{e}_{\mathbf{j}_1} ... \mathbf{e}_{\mathbf{j}_l} \otimes d\frac{1}{p!} A^{\mathbf{j}_1 \cdots \mathbf{j}_l}_{\mathbf{i}_1 \cdots \mathbf{i}_p} \theta^{\mathbf{i}_1} \wedge ... \wedge \theta^{\mathbf{i}_p}.
\end{aligned} \tag{27}$$

We have the important proposition.

Proposition 12 *Let* $A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ *and* $B \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$. *Then,*

$$d[A, B] = [dA, B] + (-1)^p [A, dB]. \tag{28}$$

Proof. The proof is a simple calculation, left to the reader. ■

We now define the exterior covariant differential operator (*EXCD)* $\mathbf{D}$ and the *extended* covariant derivative (*ECD*) $\mathbf{D}_{\mathbf{e_r}}$ acting on a Clifford valued form $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell\,(TM) \otimes \bigwedge^p T^*M$, as follows.

[8] The result printed in the original printed version is (unfortunately) wrong. However (fortunately) except for details, it does not change any of the conclusions.

3.1 Exterior Covariant Differential of Clifford Valued Forms

Definition 13 *The exterior covariant differential of $\mathcal{A}$ is the mapping :*

$$\begin{aligned}
&\mathbf{D}\colon \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \rightarrow \sec[(\bigwedge^l TM \otimes \bigwedge^p T^*M) \otimes_\wedge \bigwedge^1 T^*M] \\
&\quad \subset \sec \bigwedge^l TM \otimes \bigwedge^{p+1} T^*M, \\
&\mathbf{D}\mathcal{A} = d\mathcal{A} + \frac{p}{2}[\boldsymbol{\omega}, \mathcal{A}], \text{ if } \mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M, \ l, p \geq 1.
\end{aligned} \tag{29}$$

Proposition 14 *Let $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$, $\mathcal{B} \in \sec \bigwedge^m TM \otimes \bigwedge^q T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$. Then, the exterior differential satisfies*

$$\begin{aligned}
\mathbf{D}(\mathcal{A} \otimes_\wedge \mathcal{B}) =& \mathbf{D}\mathcal{A} \otimes_\wedge \mathcal{B} + (-1)^p \mathcal{A} \otimes_\wedge \mathbf{D}\mathcal{B} \\
&+ q[\boldsymbol{\omega}, \mathcal{A}] \otimes_\wedge \mathcal{B} + (-1)^p p \mathcal{A} \otimes_\wedge [\boldsymbol{\omega}, \mathcal{B}].
\end{aligned} \tag{30}$$

Proof. It follows directly from the definition if we take into account the properties of the product $\otimes_\wedge$ and Eq.(26).[9] ■

3.2 Extended Covariant Derivative of Clifford Valued Forms

Definition 15 *The extended covariant derivative operator is the mapping*

$$\mathbf{D}_{e_\mathbf{r}} : \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \rightarrow \sec \bigwedge^l TM \otimes \bigwedge^p T^*M,$$

such that for any $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$, l, $p \geq 1$, we have

$$\mathbf{D}\mathcal{A} = (\mathbf{D}_{\mathbf{e_r}}\mathcal{A}) \otimes_\wedge \theta^r. \tag{31}$$

We can immediately verify that

$$\mathbf{D}_{\mathbf{e_r}}\mathcal{A} = \partial_{e_\mathbf{r}}\mathcal{A} + \frac{p}{2}[\boldsymbol{\omega}_\mathbf{r}, \mathcal{A}], \tag{32}$$

and, of course, in general[10]

$$\mathbf{D}_{\mathbf{e_r}}\mathcal{A} \neq D_{e_\mathbf{r}}\mathcal{A}. \tag{33}$$

Let us write explicitly some important cases which will appear latter.

[9] Observe that $\mathbf{D}$ does not satisfy the Leibniz rule, contrary to what was stated in the original printed version. However, fortunately this does not change any conclusion.

[10] For a Clifford algebra formula for the calculation of $D_{e_\mathbf{r}}\mathcal{A}$, $\mathcal{A} \in \sec \bigwedge^p T^*M$ see Eq.(123).

3.2.1 Case $p = 1$

Let $\mathcal{A} \in \sec \bigwedge^{l} TM \otimes \bigwedge^{1} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{1} T^{*}M$. Then,

$$\mathbf{D}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\boldsymbol{\omega}, \mathcal{A}], \tag{34}$$

or

$$\mathbf{D}_{e_{\mathbf{k}}}\mathcal{A} = \partial_{e_{\mathbf{r}}}\mathcal{A} + \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{k}}, \mathcal{A}]. \tag{35}$$

3.2.2 Case $p = 2$

Let $\mathcal{F} \in \sec \bigwedge^{l} TM \otimes \bigwedge^{2} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{2} T^{*}M$. Then,

$$\mathbf{D}\mathcal{F} = d\mathcal{F} + [\boldsymbol{\omega}, \mathcal{F}], \tag{36}$$

$$\mathbf{D}_{e_{\mathbf{r}}}\mathcal{F} = \partial_{e_{\mathbf{r}}}\mathcal{F} + [\boldsymbol{\omega}_{\mathbf{r}}, \mathcal{F}]. \tag{37}$$

3.3 Cartan Exterior Differential

Recall that [26] *Cartan* defined the exterior covariant differential of $\mathfrak{C} = e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}} \in \sec \bigwedge^{1} TM \otimes \bigwedge^{p} T^{*}M$ as a mapping

$$\begin{aligned} &\mathbf{D}^{c} \colon \bigwedge^{1} TM \otimes \bigwedge^{p} T^{*}M \longrightarrow \bigwedge^{1} TM \otimes \bigwedge^{p+1} T^{*}M, \\ &\mathbf{D}^{c}\mathfrak{C} = \mathbf{D}^{c}(e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}}) = e_{\mathbf{i}} \otimes d\mathfrak{C}^{\mathbf{i}} + \mathbf{D}^{c}e_{\mathbf{i}} \wedge \mathfrak{C}^{\mathbf{i}}, \\ &\mathbf{D}^{c}e_{\mathbf{j}} = (D_{e_{\mathbf{k}}}e_{\mathbf{j}})\theta^{\mathbf{k}} \end{aligned} \tag{38}$$

which in view of Eq.(27) and Eq.(17) can be written as

$$\mathbf{D}^{c}\mathfrak{C} = \mathbf{D}^{c}(e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}}) = d\mathfrak{C} + \frac{1}{2}[\boldsymbol{\omega}, \mathfrak{C}]. \tag{39}$$

So, we have, for $p > 1$, the following relation between the exterior covariant differential $\mathbf{D}$ and Cartan's exterior differential $(p > 1)$

$$\mathbf{D}\mathfrak{C} = \mathbf{D}^{c}\mathfrak{C} + \frac{p-1}{2}[\boldsymbol{\omega}, \mathfrak{C}]. \tag{40}$$

Note moreover that when $\mathfrak{C}^{(1)} = e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}} \in \sec \bigwedge^{1} TM \otimes \bigwedge^{1} T^{*}M$, we have

$$\mathbf{D}\mathfrak{C}^{(1)} = \mathbf{D}^{c}\mathfrak{C}^{(1)}. \tag{41}$$

We end this section with two observations:

(i) There are other approaches to the concept of exterior covariant differential acting on sections of a vector bundle $E \otimes \bigwedge^{p} T^{*}M$ and also in sections of $\mathrm{end}(E)$

$\otimes\bigwedge^p T^*M$, as e.g., in [2, 3, 26, 28, 44, 46, 59]. Not all are completely equivalent among themselves and to the one presented above. Our definitions, we think, have the merit of mimicking coherently the pullback under a local section of the covariant differential acting on sections of vector bundles associated to a given principal bundle as used in gauge theories. Indeed, this consistence will be checked in several situations below.

(ii) Some authors, e.g., [3, 62] find convenient to introduce the concept of *exterior covariant derivative* of *indexed* p-forms, which are objects like the curvature 2-forms (see below) *but not* the connection 1-forms introduced above. We do not use such concept in this paper.

3.4 Torsion and Curvature

Let $\boldsymbol{\theta} = e_\mu dx^\mu = \mathbf{e_a}\theta^{\mathbf{a}} \in \sec\bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\bigwedge^1 T^*M$ and $\boldsymbol{\omega} = \frac{1}{2}\left(\omega_{\mathbf{a}}^{\mathbf{bc}}\mathbf{e_b}\wedge\mathbf{e_c}\right)\otimes\theta^{\mathbf{a}} \equiv \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}}\mathbf{e_b}\mathbf{e_c}\theta^{\mathbf{a}} \in \sec\bigwedge^2 M\otimes\bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\bigwedge^1 T^*M$ be respectively the *representatives* of a soldering form and a connection on the *basis manifold.* Then, following the standard procedure [31], the *torsion* of the connection and the *curvature* of the connection on the basis manifold are defined by

$$\boldsymbol{\Theta} = \mathbf{D}\boldsymbol{\theta} \in \sec\bigwedge^1 TM\otimes\bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\bigwedge^2 T^*M, \tag{42}$$

and

$$\mathcal{R} = \mathbf{D}\boldsymbol{\omega} \in \sec\bigwedge^2 M\otimes\bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\bigwedge^2 T^*M. \tag{43}$$

We now calculate $\boldsymbol{\Theta}$ and $\mathbf{D}\mathcal{R}$. We have,

$$\mathbf{D}\boldsymbol{\theta} = \mathbf{D}(\mathbf{e_a}\theta^{\mathbf{a}}) = \mathbf{e_a}d\theta^{\mathbf{a}} + \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{a}},\mathbf{e_d}]\theta^{\mathbf{a}}\wedge\theta^{\mathbf{d}} \tag{44}$$

and since $\frac{1}{2}[\boldsymbol{\omega}_{\mathbf{a}},\mathbf{e_d}] = -\mathbf{e_d}\lrcorner\boldsymbol{\omega}_{\mathbf{a}} = \omega_{\mathbf{ad}}^{\mathbf{c}}\mathbf{e_c}$ we have

$$\mathbf{D}(\mathbf{e_a}\theta^{\mathbf{a}}) = \mathbf{e_a}[d\theta^{\mathbf{a}} + \omega_{\mathbf{bd}}^{\mathbf{a}}\theta^{\mathbf{b}}\wedge\theta^{\mathbf{d}}] = \mathbf{e_a}\boldsymbol{\Theta}^{\mathbf{a}}, \tag{45}$$

and we recognize

$$\boldsymbol{\Theta}^{\mathbf{a}} = d\theta^{\mathbf{a}} + \omega_{\mathbf{bd}}^{\mathbf{a}}\theta^{\mathbf{b}}\wedge\theta^{\mathbf{d}}, \tag{46}$$

as *Cartan's first structure equation.*

For a torsion free connection, the torsion 2-forms $\boldsymbol{\Theta}^{\mathbf{a}} = 0$, and it follows that $\boldsymbol{\Theta} = 0$. A metric compatible connection $\boldsymbol{\omega}$ (for which $D_{\mathbf{e_a}}g = 0, \mathbf{a} = 0,1,2,3$) satisfying $\boldsymbol{\Theta}^{\mathbf{a}} = 0$ is called a Levi-Civita connection. In the remaining of this paper we *restrict* ourself to that case.

Now, according to Eq.(29) we have,

$$\mathbf{D}\mathcal{R} = d\mathcal{R} + [\boldsymbol{\omega},\mathcal{R}]. \tag{47}$$

Now, taking into account that

$$\mathcal{R} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}], \tag{48}$$

and that from Eqs.(23).(24) and (28) it follows that

$$\begin{aligned} d[\boldsymbol{\omega}, \boldsymbol{\omega}] &= [d\boldsymbol{\omega}, \boldsymbol{\omega}] - [\boldsymbol{\omega}, d\boldsymbol{\omega}], \\ [d\boldsymbol{\omega}, \boldsymbol{\omega}] &= -[\boldsymbol{\omega}, d\boldsymbol{\omega}], \\ [[\boldsymbol{\omega}, \boldsymbol{\omega}], \boldsymbol{\omega}] &= 0, \end{aligned} \tag{49}$$

we have immediately

$$\mathbf{D}\mathcal{R} = d\mathcal{R} + [\boldsymbol{\omega}, \mathcal{R}] = 0. \tag{50}$$

Eq.(50) is known as the *Bianchi identity.*

Note that, since $\{\mathbf{e_a}\}$ is an orthonormal frame we can write:

$$\begin{aligned} \mathcal{R} &= \frac{1}{4} R^{\mathbf{ab}}_{\mu\nu} \mathbf{e_a} \wedge \mathbf{e_b} \otimes (dx^\mu \wedge dx^\nu) \\ &\equiv \frac{1}{4} \mathcal{R}^{\mathbf{ab}}_{\mathbf{cd}} \mathbf{e_a} \mathbf{e_b} \otimes \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}} = \frac{1}{4} R^{\alpha\beta}_{\rho\sigma} e_\alpha e_\beta \otimes dx^\rho \wedge dx^\sigma \\ &= \frac{1}{4} R_{\mu\nu\rho\sigma} e^\mu e^\nu \otimes dx^\rho \wedge dx^\sigma, \end{aligned} \tag{51}$$

where $R_{\mu\nu\rho\sigma}$ are the components of the curvature tensor, also known in differential geometry as the Riemann tensor. We recall the well known symmetries

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma}, \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}, \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu}. \end{aligned} \tag{52}$$

We also write Eq.(51) as

$$\begin{aligned} \mathcal{R} &= \frac{1}{4} R^{\mathbf{ab}}_{\mathbf{cd}} \mathbf{e_a} \mathbf{e_b} \otimes (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) = \frac{1}{2} \mathbf{R}_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} \mathcal{R}^{\mathbf{a}}_{\mathbf{b}}\ \mathbf{e_a} \mathbf{e^b}, \end{aligned} \tag{53}$$

with

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= \frac{1}{2} R^{\mathbf{ab}}_{\mu\nu} \mathbf{e_a} \mathbf{e_b} = \frac{1}{2} R^{\mathbf{ab}}_{\mu\nu} \mathbf{e_a} \wedge \mathbf{e_b} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM), \\ \mathcal{R}^{\mathbf{ab}} &= \frac{1}{2} R^{\mathbf{ab}}_{\mu\nu} dx^\mu \wedge dx^\nu \in \sec \bigwedge^2 T^*M, \end{aligned} \tag{54}$$

where $\mathbf{R}_{\mu\nu}$ will be called curvature bivectors and the $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}$ are called after Cartan the curvature 2-forms. The $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}$ satisfy *Cartan's second structure equation*

$$\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} = d\boldsymbol{\omega}^{\mathbf{a}}_{\mathbf{b}} + \boldsymbol{\omega}^{\mathbf{a}}_{\mathbf{c}} \wedge \boldsymbol{\omega}^{\mathbf{c}}_{\mathbf{d}}, \tag{55}$$

which follows calculating $d\mathcal{R}$ from Eq.(48). Now, we can also write,

$$\begin{aligned}
\mathbf{D}\mathcal{R} &= d\mathcal{R} + [\boldsymbol{\omega}, \mathcal{R}] \\
&= \frac{1}{2}\{d(\frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e_a}\mathbf{e_b}dx^{\mu}\wedge dx^{\nu}) + [\boldsymbol{\omega}_{\rho}, \mathbf{R}_{\mu\nu}]\}dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu} \\
&= \frac{1}{2}\{\partial_{\rho}\mathbf{R}_{\mu\nu} + [\boldsymbol{\omega}_{\rho}, \mathbf{R}_{\mu\nu}]\}dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu} \\
&= \frac{1}{2}\mathbf{D}_{e_{\rho}}\mathbf{R}_{\mu\nu}dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu} \\
&= \frac{1}{3!}\left(\mathbf{D}_{e_{\rho}}\mathbf{R}_{\mu\nu} + \mathbf{D}_{e_{\mu}}\mathbf{R}_{\nu\rho} + \mathbf{D}_{e_{\nu}}\mathbf{R}_{\rho\mu}\right)dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu} = 0,
\end{aligned} \tag{56}$$

from where it follows that

$$\mathbf{D}_{e_{\rho}}\mathbf{R}_{\mu\nu} + \mathbf{D}_{e_{\mu}}\mathbf{R}_{\nu\rho} + \mathbf{D}_{e_{\nu}}\mathbf{R}_{\rho\mu} = 0. \tag{57}$$

Remark 16 *Eq.(57) is called in Physics textbooks on gauge theories (see, e.g., [44, 50]) Bianchi identity. Note that physicists call the extended covariant derivative operator*

$$\mathbf{D}_{e_{\rho}} \equiv \mathbf{D}_{\rho} = \partial_{\rho} + [\boldsymbol{\omega}_{\rho},], \tag{58}$$

acting on the curvature bivectors as the 'covariant derivative'. Note however that, as detailed above, this operator is not the usual covariant derivative operator $D_{\mathbf{e}_a}$ acting on sections of the tensor bundle.

We now find the explicit expression for the curvature bivectors $\mathbf{R}_{\mu\nu}$ in terms of the connections bivectors $\boldsymbol{\omega}_{\mu} = \boldsymbol{\omega}(e_{\mu})$, which will be used latter. First recall that by definition

$$\mathbf{R}_{\mu\nu} = \mathcal{R}(e_{\mu}, e_{\nu}) = -\mathcal{R}(e_{\nu}, e_{\mu}) = -\mathbf{R}_{\mu\nu}. \tag{59}$$

Now, observe that using Eqs.(23), (24) and (28) we can easily show that

$$\begin{aligned}
[\boldsymbol{\omega}, \boldsymbol{\omega}](e_{\mu}, e_{\nu}) &= 2[\boldsymbol{\omega}(e_{\mu}), \boldsymbol{\omega}(e_{\nu})] \\
&= 2[\boldsymbol{\omega}_{\mu}, \boldsymbol{\omega}_{\nu}].
\end{aligned} \tag{60}$$

Using Eqs. (48), (59) and (60) we get

$$\mathbf{R}_{\mu\nu} = \partial_{\mu}\boldsymbol{\omega}_{v} - \partial_{v}\boldsymbol{\omega}_{\mu} + [\boldsymbol{\omega}_{\mu}, \boldsymbol{\omega}_{\nu}]. \tag{61}$$

3.5 Some Useful Formulas

Let $A \in \sec \bigwedge^p TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ and $\mathcal{R}$ the curvature of the connection as defined in Eq.(43). Then[11] as a detailed calculation can show,

$$\mathbf{D}^2 A = \frac{1}{4}[\mathcal{R}, A] + \frac{1}{4}[d\boldsymbol{\omega}, A]. \tag{62}$$

[11] In the original printed version it is unfortunately written $\mathbf{D}^2 A = \frac{1}{2}[\mathcal{R}, A]$.

Also, we can show using the previous result that if $\mathcal{A} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ it holds[12]

$$\mathbf{D}^2\mathcal{A} = \frac{1}{4}[\mathcal{R}, \mathcal{A}] + \frac{1}{2}[\boldsymbol{\omega}, dA]. \tag{63}$$

4 General Relativity as a $Sl\,(2,\mathbb{C})$ Gauge Theory

4.1 The Nonhomogeneous Field Equations

The analogy of the fields $\mathbf{R}_{\mu\nu} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e_a}\mathbf{e_b} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e_a} \wedge \mathbf{e_b} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM)$ with the gauge fields of particle fields is so appealing that it is irresistible to propose some kind of a $Sl(2,\mathbb{C})$ formulation for the gravitational field. And indeed this has already been done, and the interested reader may consult, e.g., [6, 37]. Here, we observe that despite the similarities, the gauge theories of particle physics are in general formulated in flat Minkowski spacetime and the theory here must be for a field on a general Lorentzian spacetime. This introduces additional complications, but it is not our purpose to discuss that issue with all attention it deserves here. Indeed, for our purposes in this paper we will need only to recall some facts.

To start, recall that in gauge theories besides the homogenous field equations given by Bianchi's identities, we also have the nonhomogeneous field equation. This equation, in analogy to the nonhomogeneous equation for the electromagnetic field (see Eq.(134) in Appendix A) is written here as

$$\mathbf{D}\star\mathcal{R} = d\star\mathcal{R} + [\boldsymbol{\omega}, \star\mathcal{R}] = -\star\mathcal{J}, \tag{64}$$

where the $\mathcal{J} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ is a 'current', which, if the theory is to be one equivalent to General Relativity, must be in some way related with the energy momentum tensor in Einstein theory. In order to write from this equation an equation for the curvature bivectors, it is very useful to imagine that $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(T^*M)$, the Clifford bundle of differential forms, for in that case the powerful calculus described in the Appendix A can be used. So, we write:

$$\begin{aligned}
&\boldsymbol{\omega} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \mathcal{C}\ell(T^*M),\\
&\mathcal{R} = \mathbf{D}\boldsymbol{\omega} \in \sec \bigwedge^2 TM \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \mathcal{C}\ell(T^*M)\\
&\mathcal{J} = \mathbf{J}_\nu \otimes \theta^\nu \equiv \mathbf{J}_\nu\theta^\nu \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \mathcal{C}\ell(T^*M).
\end{aligned} \tag{65}$$

Now, using Eq.(121) for the Hodge star operator given in the Appendix A.3 and the relation between the operators $d = \boldsymbol{\partial}\wedge$ and $\delta = -\boldsymbol{\partial}\lrcorner$ (Appendix A5) we can write

$$d\star\mathcal{R} = -\theta^5(-\boldsymbol{\partial}\lrcorner\mathcal{R}) = -\star(\boldsymbol{\partial}\lrcorner\mathcal{R}) = -\star((\partial_\mu \mathbf{R}_\nu^\mu)\theta^\nu). \tag{66}$$

[12]In the printed version it is unfortunately written that $\mathbf{D}^2\mathcal{A} = \frac{1}{2}[\mathcal{R}, \mathcal{A}]$.

Also,

$$
\begin{aligned}
[\boldsymbol{\omega}, \star\mathcal{R}] &= [\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \star(\theta^\alpha \wedge \theta^\beta) \\
&= -[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \theta^5(\theta^\alpha \wedge \theta^\beta) \\
&= -\frac{1}{2}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu\theta^5(\theta^\alpha \wedge \theta^\beta) + \theta^5(\theta^\alpha \wedge \theta^\beta)\theta^\mu\} \\
&= \frac{\theta^5}{2}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu(\theta^\alpha \wedge \theta^\beta) - (\theta^\alpha \wedge \theta^\beta)\theta^\mu\} \\
&= \theta^5[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu \lrcorner (\theta^\alpha \wedge \theta^\beta) \\
&= -2 \star ([\boldsymbol{\omega}_\mu, \mathbf{R}^\mu_\beta]\theta^\beta. \qquad (67)
\end{aligned}
$$

Using Eqs.(64-67) we get[13]

$$
\partial_\mu \mathbf{R}^\mu_\nu + 2[\boldsymbol{\omega}_\mu, \mathbf{R}^\mu_\nu] = \mathbf{D}_{e_\mu}\mathbf{R}^\mu_\nu = \mathbf{J}_\nu. \qquad (68)
$$

So, the gauge theory of gravitation has as field equations the Eq.(68), the nonhomogeneous field equations, and Eq. (57) the homogeneous field equations (which is Bianchi's identity). We summarize that equations, as

$$
\mathbf{D}_{e_\mu}\mathbf{R}^\mu_\nu = \mathbf{J}_\nu, \qquad \mathbf{D}_{e_\rho}\mathbf{R}_{\mu\nu} + \mathbf{D}_{e_\mu}\mathbf{R}_{\nu\rho} + \mathbf{D}_{e_\nu}\mathbf{R}_{\rho\mu} = 0. \qquad (69)
$$

Eqs.(69) which looks like Maxwell equations, must, of course, be compatible with Einstein's equations, which may be eventually used to determine $\mathbf{R}^\mu_\nu, \boldsymbol{\omega}_\mu$ and $\mathbf{J}_\nu$.

5 Another Set of Maxwell-Like Nonhomogeneous Equations for Einstein Theory

We now show, e.g., how a special combination of the $\mathbf{R}^{\mathbf{a}}_{\mathbf{b}}$ are directly related with a combination of products of the energy-momentum 1-vectors $T_{\mathbf{a}}$ and the tetrad fields $\mathbf{e}_{\mathbf{a}}$ (see Eq.(72) below) in Einstein theory. In order to do that, we recall that Einstein's equations can be written in components in an orthonormal basis as

$$
R_{\mathbf{ab}} - \frac{1}{2}\eta_{\mathbf{ab}}R = T_{\mathbf{ab}}, \qquad (70)
$$

where $R_{\mathbf{ab}} = R_{\mathbf{ba}}$ are the components of the Ricci tensor ($R_{\mathbf{ab}} = R_{\mathbf{a}\ \mathbf{bc}}^{\ \mathbf{c}}$), $T_{\mathbf{ab}}$ are the components of the energy-momentum tensor of matter fields and $R = \eta_{\mathbf{ab}}R^{\mathbf{ab}}$ is the curvature scalar. We next introduce[14] the *Ricci 1-vectors* and the *energy-momentum 1-vectors* by

$$
R_{\mathbf{a}} = R_{\mathbf{ab}}\mathbf{e}^{\mathbf{b}} \in \sec \bigwedge\nolimits^1 TM \hookrightarrow \mathcal{C}\ell(TM), \qquad (71)
$$

$$
T_{\mathbf{a}} = T_{\mathbf{ab}}\mathbf{e}^{\mathbf{b}} \in \sec \bigwedge\nolimits^1 TM \hookrightarrow \mathcal{C}\ell(TM). \qquad (72)
$$

[13]Recall that $\mathbf{J}_\nu \in \sec \bigwedge^2 TM \hookrightarrow \sec \mathcal{C}\ell(TM)$.

[14]Ricci 1-form fields appear naturally when we formulate Einstein's equations in temrs of tetrad fields. See Appendix B.

We have that

$$R_{\mathbf{a}} = -\mathbf{e}^{\mathbf{b}} \lrcorner \mathbf{R}_{\mathbf{ab}}. \tag{73}$$

Now, multiplying Eq.(70) on the right by $\mathbf{e}^{\mathbf{b}}$ we get

$$R_{\mathbf{a}} - \frac{1}{2} R \mathbf{e}_{\mathbf{a}} = T_{\mathbf{a}}. \tag{74}$$

Multiplying Eq.(74) first on the right by $\mathbf{e}_{\mathbf{b}}$ and then on the left by $\mathbf{e}_{\mathbf{b}}$ and making the difference of the resulting equations we get

$$(-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}})\, \mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\, (-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}}) - \frac{1}{2} R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}) = (T_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}} T_{\mathbf{a}}). \tag{75}$$

Defining

$$\begin{aligned}
\mathcal{F}_{\mathbf{ab}} &= (-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}})\, \mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\, (-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}}) - \frac{1}{2} R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}) \\
&= \frac{1}{2}(R_{\mathbf{ac}}\mathbf{e}^{\mathbf{c}}\mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{b}}\mathbf{e}^{\mathbf{c}} R_{\mathbf{ac}} - \mathbf{e}^{\mathbf{c}} R_{\mathbf{ac}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}} R_{\mathbf{ac}}\mathbf{e}^{\mathbf{c}}) - \frac{1}{2} R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}})
\end{aligned} \tag{76}$$

and

$$\mathcal{J}_{\mathbf{b}} = D_{\mathbf{e}_{\mathbf{a}}}(T^{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}} T^{\mathbf{a}}), \tag{77}$$

we have[15]

$$D_{\mathbf{e}_{\mathbf{a}}} \mathcal{F}^{\mathbf{a}}_{\mathbf{b}} = \mathcal{J}_{\mathbf{b}}. \tag{78}$$

It is quite obvious that in a coordinate chart $\langle x^{\mu} \rangle$ covering an open set $U \subset M$ we can write

$$D_{e_{\rho}} \mathcal{F}^{\rho}_{\beta} = \mathcal{J}_{\beta}, \tag{79}$$

with $\mathcal{F}^{\rho}_{\beta} = g^{\rho\alpha} \mathcal{F}_{\alpha\beta}$

$$\mathcal{F}_{\alpha\beta} = (-e^{\gamma} \lrcorner \mathbf{R}_{\alpha\gamma})\, e_{\beta} - e_{\beta}\, (-e^{\gamma} \lrcorner \mathbf{R}_{\alpha\gamma}) - \frac{1}{2} R(e_{\alpha} e_{\beta} - e_{\beta} e_{\alpha}) \tag{80}$$

$$\mathcal{J}_{\beta} = D_{e_{\rho}}(T^{\rho} e_{\beta} - e^{\rho} T_{\beta}). \tag{81}$$

Remark 17 *Eq.(78) (or Eq.(79)) is a set Maxwell-like nonhomogeneous equations. It looks like the nonhomogeneous classical Maxwell equations when that equations are written in components, but Eq.(79) is only a new way of writing the equation of the nonhomogeneous field equations in the* $Sl(2,\mathbb{C})$ *like gauge theory version of Einstein's theory, discussed in the previous section. In particular, recall that any one of the six* $\mathcal{F}^{\rho}_{\beta} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM)$. *Or, in words, each one of the* $\mathcal{F}^{\rho}_{\beta}$ *is a bivector field, not a set of scalars which are components of a 2-form, as is the case in Maxwell theory. Also, recall that according to Eq.(81) each one of the four* $\mathcal{J}_{\beta} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM)$.

[15] Note that we could also produce another Maxwell-like equation, by using the extended covariant derivative operator in the definition of the current, i.e., we can put $\mathcal{J}_{\mathbf{b}} = D_{\mathbf{e}_{\mathbf{a}}}(T^{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}} T^{\mathbf{a}})$, and in that case we obtain $\mathbf{D}_{\mathbf{e}_{\mathbf{a}}} \mathcal{F}^{\mathbf{a}}_{\mathbf{b}} = \mathcal{J}_{\mathbf{b}}$.

From Eq.(78) it is not obvious that we must have $\mathcal{F}_{\mathbf{ab}} = 0$ in vacuum, however that is exactly what happens if we take into account Eq.(76) which defines that object. Moreover, $\mathcal{F}_{\mathbf{ab}} = 0$ does not imply that the curvature bivectors $\mathbf{R}_{\mathbf{ab}}$ are null in vacuum. Indeed, in that case, Eq.(76) implies only the identity (valid *only* in vacuum)

$$(\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}})\,\mathbf{e}_{\mathbf{b}} = (\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{bc}})\,\mathbf{e}_{\mathbf{a}}. \tag{82}$$

Moreover, recalling definition (Eq.(54)) we have

$$\mathbf{R}_{\mathbf{ab}} = R_{\mathbf{abcd}} \mathbf{e}^{\mathbf{c}} \mathbf{e}^{\mathbf{d}}, \tag{83}$$

and we see that the $\mathbf{R}_{\mathbf{ab}}$ are zero only if the Riemann tensor is null which is not the case in any non trivial general relativistic model.

The important fact that we want to emphasize here is that although eventually interesting, Eq.(78) does not seem (according to our opinion) to contain anything new in it. More precisely, all information given by that equation is already contained in the original Einstein's equation, for indeed it has been obtained from it by simple algebraic manipulations. We state again: According to our view terms like

$$\begin{aligned}
\mathcal{F}_{\mathbf{ab}} &= \frac{1}{2}(R_{\mathbf{ac}}\mathbf{e}^{\mathbf{c}}\mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{b}}\mathbf{e}^{\mathbf{c}}R_{\mathbf{ac}} - \mathbf{e}^{\mathbf{c}}R_{\mathbf{ac}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}R_{\mathbf{ac}}\mathbf{e}^{\mathbf{c}}) - \frac{1}{2}R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}),\\
\mathfrak{R}_{\mathbf{ab}} &= (T_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}T_{\mathbf{a}}) - \frac{1}{2}R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}),\\
\mathbf{F}_{\mathbf{ab}} &= \frac{1}{2}R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}),
\end{aligned} \tag{84}$$

are pure gravitational objects. We cannot see any relationship of any one of these objects with the ones appearing in Maxwell theory. Of course, these objects may eventually be used to formulate interesting equations, like Eq.(78) which are equivalent to Einstein's field equations, but this fact does not seem to us to point to any new Physics.[16] Even more, from the mathematical point of view, to find solutions to the new Eq.(78) is certainly as hard as to find solutions to the original Einstein equations.

5.1 $Sl\,(2,\mathbb{C})$ Gauge Theory and Sachs Antisymmetric Equation

We discuss in this subsection yet another algebraic exercise. First recall that in section 2 of [45] we define the paravector fields,

$$\mathbf{q}_{\mathbf{a}} = \mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{0}} = \boldsymbol{\sigma}_{\mathbf{a}}, \quad \check{\mathbf{q}}_{\mathbf{a}} = (-\boldsymbol{\sigma}_{\mathbf{0}}, \boldsymbol{\sigma}_{\mathbf{i}}), \quad \boldsymbol{\sigma}_{\mathbf{0}} = 1.$$

[16] Note that $\mathbf{F}_{\mathbf{ab}}$ differs from a factor, namely R from the $\mathbf{F}'_{\mathbf{ab}}$ give by Eq.(70) in [45].

Recall that[17]

$$
\begin{aligned}
[D_{e_\rho}, D_{e_\lambda}]e_\mu &= R_{\mu\ \rho\lambda}^{\ \alpha} e_\alpha = -R_{\alpha\mu\rho\lambda}e^\alpha = R_{\mu\alpha\rho\lambda}e^\alpha, \\
R_{\mu\ \rho\lambda}^{\ \alpha} &= \mathcal{R}(e_\mu, \theta^\alpha, e_\rho, e_\lambda).
\end{aligned} \tag{85}
$$

Then a simple calculation shows that

$$
[D_{e_\rho}, D_{e_\lambda}]e_\mu = e_\mu \lrcorner \mathbf{R}_{\rho\lambda} = -\mathbf{R}_{\rho\lambda} \llcorner e_\mu, \tag{86}
$$

$$
R_{\mu\alpha\rho\lambda}e^\alpha = \frac{1}{2}(e_\mu \mathbf{R}_{\rho\lambda} - \mathbf{R}_{\rho\lambda} e_\mu). \tag{87}
$$

Multiplying Eq.(87) on the left by $\mathbf{e_0}$ we get, recalling that $\boldsymbol{\omega}_{\mathbf{e_a}}^\dagger = -\mathbf{e^0}\boldsymbol{\omega}_{\mathbf{e_a}}\mathbf{e^0}$ (Eq.(79) in [45]) we get

$$
R_{\mu\alpha\rho\lambda}\mathbf{q}^\alpha = \frac{1}{2}(\mathbf{q}_\mu \mathbf{R}_{\rho\lambda}^\dagger + \mathbf{R}_{\rho\lambda}\mathbf{q}_\mu). \tag{88}
$$

Now, to derive Sachs[18] Eq.(6.50a) all we need to do is to multiply Eq.(75) on the right by $\mathbf{e^0}$ and perform some algebraic manipulations. We then get (with *our* normalization) for the equivalent of Einstein's equations using the paravector fields and a coordinate chart $\langle x^\mu \rangle$ covering an open set $U \subset M$, the following equation

$$
\mathbf{R}_{\rho\lambda}\mathbf{q}^\lambda + \mathbf{q}^\lambda \mathbf{R}_{\rho\lambda}^\dagger + R\mathbf{q}_\rho = 2\mathbf{T}_\rho. \tag{89}
$$

For the Hermitian conjugate we have

$$
-\mathbf{R}_{\rho\lambda}^\dagger \check{\mathbf{q}}^\lambda - \check{\mathbf{q}}^\lambda \mathbf{R}_{\rho\lambda} + R\check{\mathbf{q}}_\rho = 2\check{\mathbf{T}}_\rho, \tag{90}
$$

where as above, the $\mathbf{R}_{\rho\lambda}$ are the curvature bivectors given by Eq.(61) and

$$
\mathbf{T}_\rho = T_\rho^\mu \mathbf{q}_\mu \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM). \tag{91}
$$

After that, we multiply Eq.(89) on the right by $\check{\mathbf{q}}_\gamma$ and Eq.(90) on the left by $\mathbf{q}_\gamma$ ending with two new equations. If we sum them, we get a 'symmetric' equation[19] completely equivalent to Einstein's equation (from where we started). If we make the difference of the equations we get an antisymmetric equation. The antisymmetric equation can be written, introducing

$$
\begin{aligned}
\mathbb{F}_{\rho\gamma} &= \frac{1}{2}(\mathbf{R}_{\rho\lambda}\mathbf{q}^\lambda\check{\mathbf{q}}_\gamma + \mathbf{q}_\gamma\check{\mathbf{q}}^\lambda\mathbf{R}_{\rho\lambda} + \mathbf{q}^\lambda\mathbf{R}_{\rho\lambda}^\dagger\check{\mathbf{q}}_\gamma + \mathbf{q}_\gamma\mathbf{R}_{\rho\lambda}^\dagger\check{\mathbf{q}}^\lambda) \\
&+ \frac{1}{2}R(\mathbf{q}_\rho\check{\mathbf{q}}_\gamma - \mathbf{q}_\gamma\check{\mathbf{q}}_\rho)
\end{aligned} \tag{92}
$$

[17] In Sachs book he wrote: $[D_{e_\rho}, D_{e_\lambda}]e_\mu = R_{\mu\ \rho\lambda}^{\ \alpha} e_\alpha = +R_{\alpha\mu\rho\lambda}e^\alpha$. This produces some changes in signals in relation to our formulas below. Our Eq.(85) agrees with the conventions in [9].

[18] Numeration is from Sachs' book [51].

[19] Eq.(6.52) in Sachs' book [51].

and

$$\mathbb{J}_\gamma = D_{e_\rho}(\mathbf{T}^\rho \check{\mathbf{q}}_\gamma - \mathbf{q}_\gamma \check{\mathbf{T}}^\rho), \tag{93}$$

as

$$D_{e_\rho}\mathbb{F}^\rho_\gamma = \mathbb{J}_\gamma. \tag{94}$$

Remark 18 *It is important to keep in mind that each one of the six $\mathbb{F}_{\rho\gamma}$ and each one of the four $\mathbb{J}_\gamma$ are not a set of scalars, but sections of $\mathcal{C}\ell^{(0)}(TM)$. Also, take notice that Eq.(94), of course, is completely equivalent to our Eq.(78). Its matrix translation in $\mathbb{C}\ell^{(0)}(M) \simeq S(M) \otimes_{\mathbb{C}} \bar{S}(M)$ gives Sachs equation (6.52-) in [51] if we take into account his different 'normalization' of the connection coefficients and the ad hoc factor with dimension of electric charge that he introduced. We cannot see at present any new information encoded in that equations which could be translated in interesting geometrical properties of the manifold, but of course, eventually someone may find that they encode such a useful information.*[20]

Using the equations, $D_{\mathbf{e_a}}\mathbf{e}_0 = 0$ and $D^{\mathbf{S}}_{e_\rho}\mathbf{q}_\mu = 0$ (respectively, Eq.(88) and Eq.(108) in [45]) and (57) we may verify that

$$D^{\mathbf{S}}_{e_\rho}\mathbb{F}_{\mu\nu} + D^{\mathbf{S}}_{e_\mu}\mathbb{F}_{\nu\rho} + D^{\mathbf{S}}_{e_\nu}\mathbb{F}_{\rho\mu} = 0, \tag{95}$$

where $D^{\mathbf{S}}_{e_\rho}$ is Sachs 'covariant' derivative that we discussed in [45]. In [52] Sachs concludes that the last equation implies that there are no magnetic monopoles in nature. Of course, his conclusion would follow from Eq.(95) only if it happened that $\mathbb{F}^\rho_\gamma$ were the components in a coordinate basis of a 2-form field $F \in \sec \bigwedge^2 T^*M$. However, this is not the case, because as already noted above, this is not the mathematical nature of the $\mathbb{F}^\rho_\gamma$. Contrary to what we stated with relation to Eq.(94) we cannot even say that Eq.(95) is really interesting, because it uses a covariant derivative operator, which, as discussed in [45] is not well justified, and in anyway $D^{\mathbf{S}}_{e_\rho} \neq D_{e_\rho}$. We cannot see any relationship of Eq.(95) with the legendary magnetic monopoles.

We thus conclude this section stating that Sachs claims in [51, 52, 53] of having produced an unified field theory of electricity and electromagnetism are not endorsed by our analysis.

[20] Anyway, it seems to us that until the written of the present paper the true mathematical nature of Sachs equations have not been understood, by people that read Sachs books and articles. To endorse our statement, we quote that in Carmeli's review([7]) of Sachs book, he did not realize that Sachs theory was indeed (as we showed above) a description in the Pauli bundle of a $Sl(2,\mathbb{C})$ gauge formulation of Einstein's theory as described in his own book [6]. Had he disclosed that fact (as we did) he probably had not written that Sachs' approach was a possible unified field theory of gravitation and electromagnetism.

6 Energy-Momentum "Conservation" in General Relativity

6.1 Einstein's Equations in terms of Superpotentials $\star S^{\mathbf{a}}$

In this section we discuss some issues and statements concerning the problem of the energy-momentum conservation in Einstein's theory, presented with several different formalisms in the literature, which according to our view are very confusing, or even wrong. To start, recall that from Eq.(64) it follows that

$$d(\star\mathcal{J}-\frac{1}{2}[\boldsymbol{\omega},\star\mathcal{R}])=0, \tag{96}$$

and we could think that this equation could be used to identify a *conservation law* for the energy momentum of matter plus the gravitational field, with $\frac{1}{2}[\boldsymbol{\omega},\star\mathcal{R}]$ describing a mathematical object related to the energy- momentum of the gravitational field. However, this is not the case, because this term (due to the presence of $\boldsymbol{\omega}$) is gauge dependent. The appearance of a gauge dependent term is a recurrent fact in all known proposed[21] formulations of a 'conservation law for energy-momentum' for Einstein theory. We discuss now some statements found in the literature based on some of that proposed 'solutions' to the problem of energy-momentum conservation in General Relativity and say why we think they are unsatisfactory. We also mention a way with which the problem could be satisfactorily solved, but which implies in a departure from the orthodox interpretation of Einstein's theory.

Now, to keep the mathematics as simple and transparent as possible, instead of working with Eq.(96), we work with a more simple (but equivalent) formulation [47, 61] of Einstein's equation where the gravitational field is described by a set of 2-forms $\star S^{\mathbf{a}}$, $\mathbf{a}=0,1,2,3$ called superpotentials. This approach will permit to identify very quickly certain objects that at first sight seems *appropriate energy-momentum* currents for the gravitational field in Einstein's theory. The calculations that follows are done in the Clifford algebra of multiforms fields $\mathcal{C}\ell(T^{*}M)$, something that, as the reader will testify, simplify considerably similar calculations done with traditional methods.

We start again with Einstein's equations given by Eq.(70), but this time we multiply on the left by $\theta^{\mathbf{b}}\in\sec\bigwedge^{1}T^{*}M\hookrightarrow\mathcal{C}\ell(T^{*}M)$ getting an equation relating the *Ricci 1-forms* $\mathcal{R}^{\mathbf{a}}=R_{\mathbf{b}}^{\mathbf{a}}\theta^{\mathbf{b}}\in\sec\bigwedge^{1}T^{*}M\hookrightarrow\mathcal{C}\ell(T^{*}M)$ with the *energy-momentum 1-forms* $\mathcal{T}^{\mathbf{a}}=T_{\mathbf{b}}^{\mathbf{a}}\theta^{\mathbf{b}}\in\sec\bigwedge^{1}T^{*}M\hookrightarrow\mathcal{C}\ell(T^{*}M)$, i.e.,

$$\mathcal{G}^{\mathbf{a}}=\mathcal{R}^{\mathbf{a}}-\frac{1}{2}R\theta^{\mathbf{a}}=\mathcal{T}^{\mathbf{a}}. \tag{97}$$

We take the dual of this equation,

$$\star\,\mathcal{G}^{\mathbf{a}}=\star\mathcal{T}^{\mathbf{a}}. \tag{98}$$

[21] At least, the ones known by the authors.

Next, we observe that [47, 61] we can write

$$\star\,\mathcal{G}^{\mathbf{a}} = -d\star\mathcal{S}^{\mathbf{a}} - \star\mathfrak{t}^{\mathbf{a}}, \tag{99}$$

where

$$\begin{aligned}
\star\mathcal{S}^{\mathbf{c}} &= \frac{1}{2}\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}}), \\
\star\mathfrak{t}^{\mathbf{c}} &= -\frac{1}{2}\boldsymbol{\omega}_{\mathbf{ab}} \wedge [\boldsymbol{\omega}^{\mathbf{c}}_{\mathbf{d}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) - \boldsymbol{\omega}^{\mathbf{b}}_{\mathbf{d}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{d}} \wedge \theta^{\mathbf{c}})].
\end{aligned} \tag{100}$$

The proof of Eq.(100) follows at once from the fact that

$$\star\,\mathcal{G}^{\mathbf{d}} = -\frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}). \tag{101}$$

Indeed, recalling the identities in Eq.(122) we can write

$$\begin{aligned}
\frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) &= -\frac{1}{2}\star[\mathcal{R}_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\
&= -\frac{1}{2}R_{\mathbf{abcd}}\star[(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\
&= -\star(\mathcal{R}^{\mathbf{d}} - \frac{1}{2}R\theta^{\mathbf{d}}).
\end{aligned} \tag{102}$$

On the other hand we have,

$$\begin{aligned}
&-2\star\mathcal{G}^{\mathbf{d}} = d\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) + \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}^{\mathbf{c}}_{\mathbf{b}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&= d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] + \boldsymbol{\omega}_{\mathbf{ab}} \wedge d\star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&+ \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}^{\mathbf{c}}_{\mathbf{b}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&= d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}^{\mathbf{a}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{p}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&- \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}^{\mathbf{b}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}}) - \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}^{\mathbf{d}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}})] \\
&+ \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}^{\mathbf{c}}_{\mathbf{b}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&= d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \boldsymbol{\omega}_{\mathbf{ab}} \wedge [\boldsymbol{\omega}^{\mathbf{d}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}}) + \boldsymbol{\omega}^{\mathbf{b}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}})] \\
&= 2(d\star\mathcal{S}^{\mathbf{d}} + \star\mathfrak{t}^{\mathbf{d}}).
\end{aligned} \tag{103}$$

Now, we can then write Einstein's equation in a very interesting, but *dangerous* form, i.e.,

$$-d\star\mathcal{S}^{\mathbf{a}} = \star\mathcal{T}^{\mathbf{a}} + \star\mathfrak{t}^{\mathbf{a}}. \tag{104}$$

In writing Einstein's equations in that way, we have associated to the gravitational field a set of 2-form fields $\star\mathcal{S}^{\mathbf{a}}$ called *superpotentials* that have as sources the currents $(\star\mathcal{T}^{\mathbf{a}} + \star\mathfrak{t}^{\mathbf{a}})$. However, superpotentials are not uniquely defined since, e.g., superpotentials $(\star\mathcal{S}^{\mathbf{a}} + \star\alpha^{\mathbf{a}})$, with $\star\alpha^{\mathbf{a}}$ closed, i.e., $d\star\alpha^{\mathbf{a}} = 0$ give the same second member for Eq.(104).

6.2 Is There Any Energy-Momentum Conservation Law in GR?

Why did we say that Eq.(104) is a dangerous one?

The reason is that (as in the case of Eq.(96)) we can be led to think that we have discovered a conservation law for the energy-momentum of matter plus gravitational field, since from Eq.(104) it follows that

$$d(\star\mathcal{T}^{\mathbf{a}} + \star\mathbf{t}^{\mathbf{a}}) = 0. \tag{105}$$

This thought however is only an example of wishful thinking, because the $\star\mathbf{t}^{\mathbf{a}}$ depends on the connection (see Eq.(100)) and thus are gauge dependent. They do not have the same tensor transformation law as the $\star\mathcal{T}^{\mathbf{a}}$. So, Stokes theorem cannot be used to derive from Eq.(105) conserved quantities that are independent of the gauge, which is clear. However, and this is less known, Stokes theorem, also cannot be used to derive conclusions that are independent of the local coordinate chart used to perform calculations [5]. In fact, the currents $\star\mathbf{t}^{\mathbf{a}}$ are nothing more than the old pseudo energy momentum tensor of Einstein in a new dress. Nonrecognition of this fact can lead to many misunderstandings. We present some of them in what follows, in order to call our readers' attention of potential errors of inference that can be done when we use sophisticated mathematical formalisms without a perfect domain of their contents.

(i) First, it is easy to see that from Eq.(98) it follows that [38]

$$\mathbf{D}^c\star\mathfrak{G} = \mathbf{D}^c\star\mathfrak{T} = 0, \tag{106}$$

where $\star\mathfrak{G} = \mathbf{e_a}\otimes\star\mathcal{G}^{\mathbf{a}} \in \sec TM \otimes \sec \bigwedge^3 T^*M$ and $\star\mathfrak{T} = \mathbf{e_a}\otimes\star\mathcal{T}^{\mathbf{a}} \in \sec TM\otimes \sec\bigwedge^3 T^*M$. Now, in [38] it is written (without proof) a 'Stokes theorem'

$$\boxed{\int_{\text{4-cube}} \mathbf{D}^c\star\mathfrak{T} = \int_{\substack{\text{3 boundary}\\ \text{of this 4-cube}}} \star\mathfrak{T}} \tag{107}$$

We searched in the literature for a proof of Eq.(107) which appears also in many other texts and scientific papers, as e.g., in [12, 65] and could find none, which we can consider as valid. The reason is simply. If expressed in details, e.g., the first member of Eq.(107) reads

$$\int_{\text{4-cube}} \mathbf{e_a}\otimes(d\star\mathcal{T}^{\mathbf{a}} + \omega^{\mathbf{a}}_{\mathbf{b}}\wedge\mathcal{T}^{\mathbf{b}}), \tag{108}$$

and it is necessary to explain what is the meaning (if any) of the integral. Since the integrand is a sum of tensor fields, this integral says that we are *summing* tensors belonging to the tensor spaces of different spacetime points. As, well known, this cannot be done in general, unless there is a way for identification

of the tensor spaces at different spacetime points. This requires, of course, the introduction of additional structure on the spacetime representing a given gravitational field, and such extra structure is lacking in Einstein's theory. We unfortunately, must conclude that Eq.(107) do not express any conservation law, for it lacks as yet, a precise mathematical meaning.[22]

In Einstein theory possible superpotentias are, of course, the $\star\mathcal{S}^{\mathbf{a}}$ that we found above (Eq.(100)), with

$$\star\,\mathcal{S}_{\mathbf{c}} = [-\frac{1}{2}\omega_{\mathbf{ab}}\lrcorner(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta_{\mathbf{c}})]\theta^{\mathbf{5}}. \tag{109}$$

Then, if we integrate Eq.(104) over a 'certain finite 3-dimensional volume', say a ball B, and use Stokes theorem we have

$$P^{\mathbf{a}} = \int\limits_{B} \star\,(\mathcal{T}^{\mathbf{a}} + \mathfrak{t}^{\mathbf{a}}) = -\int\limits_{\partial B} \star\,\mathcal{S}^{\mathbf{a}}. \tag{110}$$

In particular the energy or (*inertial mass*) of the gravitational field plus matter generating the field is defined by

$$P^{\mathbf{0}} = E = m_i = -\lim_{R\to\infty}\int\limits_{\partial B} \star\,\mathcal{S}^{\mathbf{0}} \tag{111}$$

(ii) Now, a frequent misunderstanding is the following. Suppose that in a *given* gravitational theory there exists an energy-momentum conservation law for matter plus the gravitational field expressed in the form of Eq.(105), where $\mathcal{T}^{\mathbf{a}}$ are the energy-momentum 1-forms of matter and $\mathfrak{t}^{\mathbf{a}}$ are *true*[23] energy-momentum 1-forms of the gravitational field. This means that the 3-forms $(\star\mathcal{T}^{\mathbf{a}}+\star\mathfrak{t}^{\mathbf{a}})$ are closed, i.e., they satisfy Eq.(105). Is this enough to warrant that the energy of a closed universe is zero? Well, that would be the case if starting from Eq.(105) we could jump to an equation like Eq.(104) and then to Eq.(111) (as done, e.g., in [62]). But that sequence of inferences in general cannot be done, for indeed, as it is well known, it is not the case that closed three forms are always exact. Take a closed universe with topology, say $\mathbb{R}\times S^3$. In this case $B = S^3$ and we have $\partial B = \partial S^3 = \varnothing$. Now, as it is well known (see, e.g., [43]), the third de Rham cohomology group of $\mathbb{R}\times S^3$ is $H^3\left(\mathbb{R}\times S^3\right) = H^3\left(S^3\right) = \mathbb{R}$. Since this group is non trivial it follows that in such manifold closed forms are not exact. Then from Eq.(105) it did not follow the validity of an equation analogous to Eq.(104). So, in that case an equation like Eq.(110) cannot even be written.

Despite that commentary, keep in mind that in Einstein's theory the energy of a closed universe[24], if it is justifiable to suppose that it is given by Eq.(111),

[22]Of course, if some could give a mathematical meaning to Eq.(107), we will be glad to be informed of that fact.

[23]This means that the $t^{\mathbf{a}}$ are not pseudo 1-forms, as in Einstein's theory.

[24]Note that if we suppose that the universe contains spinor fields, then it must be a spin manifold, i.e., it is parallezible according to Geroch's theorem [27].

is indeed zero, since in that theory the 3-forms $(\star\mathcal{T}^{\mathbf{a}} + \star\mathfrak{t}^{\mathbf{a}})$ are indeed exact (see Eq.(104)). This means that accepting $\mathfrak{t}^{\mathbf{a}}$ as the 'energy-momentum' 1-form fields of the gravitational field, it follows that gravitational 'energy' *must* be negative in a closed universe.

(iii) But, is the above formalism a consistent one? Given a coordinate chart $\langle x^\mu \rangle$ of the maximal atlas of M, with some algebra we can show that for a gravitational model represented by a diagonal asymptotic flat metric[25], the inertial mass $E = m_i$ is given by

$$m_i = \lim_{R\to\infty} \frac{-1}{16\pi} \int\limits_{\partial B} \frac{\partial}{\partial x^\beta}(g_{11}g_{22}g_{33}g^{\alpha\beta})d\sigma_\alpha, \tag{112}$$

where $\partial B = S^2(R)$ is a 2-sphere of radius R, $(-n_\alpha)$ is the outward unit normal and $d\sigma_\alpha = -R^2 n_\alpha dA$. If we apply Eq.(112) to calculate, e.g., the energy of the Schwarzschild spacetime[26] generate by a gravitational mass m, we expect to have one unique and unambiguous result, namely $m_i = m$.

However, as showed in details, e.g., in [5] the calculation of E depends on the spatial coordinate system naturally adapted to the reference frame $Z = \frac{1}{\sqrt{\left(1-\frac{2m}{r}\right)}}\frac{\partial}{\partial t}$, even if these coordinates produce asymptotically flat metrics. Then, even if in one given chart we may obtain as the result of the calculation $m_i = m$ there are others where the results of the calculations give $m_i \neq m$!

Moreover, note also that, as showed above, for a closed universe, Einstein's theory implies on general grounds (once we accept that the $\mathfrak{t}^a$ describes the energy-momentum distribution of the gravitational field) that $m_i = 0$. This result, it is important to quote, does not contradict the so called "positive mass theorems" of, e.g., references [56, 57, 70], because that theorems refers to the total 'energy' of an isolated system. A system of that kind is supposed to be modelled by a Lorentzian spacetime having a spacelike, asymptotically Euclidean hypersurface.[27] However, we want to emphasize here, that although the 'energy' results positive, its value is not unique, since depends on the asymptotically flat coordinates chosen to perform the calculations, as it is clear from the example of the Schwarzschild field, as we already commented above and detailed in [5].

In view of what has been presented above, it is our view that all discourses (based on Einstein's equivalence principle) concerning the use of pseudo-energy momentum tensors as *reasonable* descriptions of energy and momentum of gravitational fields in Einstein's theory are not convincing.

The fact is: there are *in general* no conservation laws of energy-momentum in General Relativity in general. And, at this point it is better to quote page 98 of Sachs&Wu[28] [54]:

---

[25] A metric is said to be asymptotically flat in given coordinates, if $g_{\mu\nu} = n_{\mu\nu}(1 + \mathrm{O}\left(r^{-k}\right))$, with $k = 2$ or $k = 1$ depending on the author. See, eg., [56, 57, 67].

[26] For a Scharzschild spacetime we have $g = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi)$.

[27] The proof also uses as hypothesis the so called energy dominance condition. [29]

[28] Note, please, that in this reference Sachs refers to R. K. Sachs and not to M. Sachs.

" As mentioned in section 3.8, conservation laws have a great predictive power. It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity. Many of the attempts to resurrect it are quite interesting; many are simply garbage."

We quote also Anderson [1]:

" In an interaction that involves the gravitational field a system can loose energy without this energy being transmitted to the gravitational field."

In General Relativity, we already said, every gravitational field is modelled (module diffeomorphisms) by a Lorentzian spacetime. In the particular case, when this spacetime structure admits a *timelike* Killing vector, we can formulate a law of energy conservation. If the spacetime admits three linearly independent *spacelike* Killing vectors, we have a law of conservation of momentum. The crucial fact to have in mind here is that a general Lorentzian spacetime, does not admits such Killing vectors in general. As one example, we quote that the popular Friedmann-Robertson-Walker expanding universes models do not admit timelike Killing vectors, in general.

At present, the authors know only one possibility of resurrecting a *trustworthy* conservation law of energy-momentum valid in all circumstances in a theory of the gravitational field that *resembles* General Relativity (in the sense of keeping Einstein's equation). It consists in reinterpreting that theory as a field theory in flat Minkowski spacetime. Theories of this kind have been proposed in the past by, e.g., Feynman [25], Schwinger [55],Thirring [60] and Weinberg [68, 69] and have been extensively studied by Logunov and collaborators [35, 36]. Another presentation of a theory of that kind, is one where the gravitational field is represented by a distortion field in Minkowski spacetime. A first attempt to such a theory using Clifford bundles has been given in [47]. Another presentation has been given in [32], but that work, which contains many interesting ideas, unfortunately contains also some equivocated statements that make (in our opinion) the theory, as originally presented by that authors invalid. This has been discussed with details in [20].

Before closing this section we observe that recently people think to have found a valid way of having a genuine energy-momentum conservation law in general relativity, by using the so-called *teleparallel* version of that theory [14]. If that is really the case will be analyzed in a sequel paper [49], where we discuss conservation laws in a general Riemann-Cartan spacetime, using Clifford bundle methods.

7 Conclusions

In this paper we introduced the concept of Clifford valued differential forms, which are sections of $\mathcal{C}\ell(TM)\otimes\bigwedge T^*M$. We showed how this theory can be used to produce a very elegant description of the theory of linear connections, where a given linear connection is represented by a bivector valued 1-form. Crucial to the program was the introduction of the notion of the exterior covariant differential and the extended covariant derivative acting on sections of $\mathcal{C}\ell(TM)\otimes\bigwedge T^*M$.

Our *natural* definitions parallel in a noticeable way the formalism of the theory of connections in a principal bundle and the covariant derivative operators acting on associate bundles to that principal bundle. We identified Cartan curvature 2-forms and *curvature bivectors*. The curvature 2-forms satisfy Cartan's second structure equation and the curvature bivectors satisfy equations in analogy with equations of gauge theories. This immediately suggest to write Einstein's theory in that formalism, something that has already been done and extensively studied in the past, but not with the methods used in this paper. However, we did not enter into the details of that theory in this paper. We only discussed the relation between the nonhomogeneous $Sl(2,\mathbb{C})$ gauge equation satisfied by the curvature bivector and some other mathematically equivalent formulations of Einstein's field equations and also we carefully analyzed the relation of nonhomogeneous $Sl(2,\mathbb{C})$ gauge equation satisfied by the curvature bivector and some interesting equations that appears in M. Sachs 'unified' field theory as described recently in [53] and originally introduced in [51]. In these books the interested reader may also find a complete list of references to Sachs papers published in the last 40 years. Next, taking profit of the mathematical methods introduced in this paper, we discussed also some issues concerning the very important problem of the energy-momentum 'conservation' in General Relativity, which has defied scientists for almost all the twenty century, since Hilbert and Einstein found the field equations of the gravitational field.

Finally, we present in Appendix A the main results of the Clifford bundle formalism, and in Appendix B we detail the description of Einstein's equations for the tetrad fields, using the Clifford bundle formalism, where some very nice operators, which have no analogy on classical differential geometry are exhibit.

Acknowledgment. Authors are grateful to Dr. Ricardo A. Mosna for very useful observations, and to a referee for very important comments.

A Clifford Bundles $\mathcal{C}\ell(T^*M)$ and $\mathcal{C}\ell(TM)$

Let $\mathcal{L} = (M, g, D, \tau_g, \uparrow)$ be a Lorentzian spacetime. This means that $(M, g, \tau_g, \uparrow)$ is a four dimensional Lorentzian manifold, time oriented by $\uparrow$ and spacetime oriented by τ_g, and in general $M \neq \mathbb{R}^4$. Also, $g \in \sec(T^*M \times T^*M)$ is a Lorentzian metric of signature (1,3). T^*M $[TM]$ is the cotangent [tangent] bundle. $T^*M = \cup_{x\in M} T^*_x M$, $TM = \cup_{x\in M} T_x M$, and $T_xM \simeq T^*_xM \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space [54]. D is the Levi-Civita connection of g, i.e., $Dg = 0$, $\mathcal{R}(D) = 0$. Also $\Theta(D) = 0$, $\mathcal{R}$ and Θ being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{C}\ell(T^*M)$ is the bundle of algebras[29] $\mathcal{C}\ell(T^*M) = \cup_{x\in M} \mathcal{C}\ell(T^*_xM)$, where $\forall x \in M, \mathcal{C}\ell(T^*_xM) = \mathbb{R}_{1,3}$, the so-called *spacetime algebra* [34]. Locally as a linear space over the real field R, $\mathcal{C}\ell(T^*_xM)$ is isomorphic to the Car-

[29] We can show using the definitions of section 5 that $\mathcal{C}\ell(T^*M)$ is a vector bundle associated with the *orthonormal frame bundle*, i.e., $\mathcal{C}\ell(M) = P_{SO_{+(1,3)}} \times_{ad} Cl_{1,3}$. Details about this construction can be found, e.g., in [39].

tan algebra $\bigwedge(T_x^*M)$ of the cotangent space and $\bigwedge T_x^*M = \sum_{k=0}^{4} \bigwedge^k T_x^*M$, where $\bigwedge^k T_x^*M$ is the $\binom{4}{k}$-dimensional space of k-forms. The Cartan bundle $\bigwedge T^*M = \cup_{x\in M} \bigwedge T_x^*M$ can then be thought [33] as "imbedded" in $\mathcal{C}\ell(T^*M)$. In this way sections of $\mathcal{C}\ell(T^*M)$ can be represented as a sum of nonhomogeneous differential forms. Let $\{\mathbf{e_a}\} \in \sec TM, (\mathbf{a} = 0,1,2,3)$ be an orthonormal basis $g(\mathbf{e_a}, \mathbf{e_b}) = \eta_{\mathbf{ab}} = \operatorname{diag}(1,-1,-1,-1)$ and let $\{\theta^{\mathbf{a}}\} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$ be the dual basis. Moreover, we denote by g^{-1} the metric in the cotangent bundle.

An analogous construction can be done for the tangent space. The corresponding Clifford bundle is denoted $\mathcal{C}\ell(TM)$ and their sections are called multivector fields. All formulas presented below for $\mathcal{C}\ell(T^*M)$ have corresponding ones in $\mathcal{C}\ell(TM)$ and this fact has been used in the text.

A.1 Clifford product, scalar contraction and exterior products

The fundamental *Clifford product*[30] (in what follows to be denoted by juxtaposition of symbols) is generated by $\theta^{\mathbf{a}}\theta^{\mathbf{b}} + \theta^{\mathbf{b}}\theta^{\mathbf{a}} = 2\eta^{\mathbf{ab}}$ and if $\mathcal{C} \in \sec \mathcal{C}\ell(T^*M)$ we have **??**

$$\mathcal{C} = s + v_{\mathbf{a}}\theta^{\mathbf{a}} + \frac{1}{2!} b_{\mathbf{cd}}\theta^{\mathbf{c}}\theta^{\mathbf{d}} + \frac{1}{3!} a_{\mathbf{abc}}\theta^{\mathbf{a}}\theta^{\mathbf{b}}\theta^{\mathbf{c}} + p\theta^{\mathbf{5}} , \tag{113}$$

where $\theta^{\mathbf{5}} = \theta^0\theta^{\mathbf{1}}\theta^{\mathbf{2}}\theta^{\mathbf{3}}$ is the volume element and $s, v_{\mathbf{a}}, b_{\mathbf{cd}}, a_{\mathbf{abc}}, p \in \sec \bigwedge^0 T^*M \subset \sec \mathcal{C}\ell(T^*M)$.

Let $A_r, \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M), B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$. For $r = s = 1$, we define the *scalar product* as follows:

For $a, b \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$,

$$a \cdot b = \frac{1}{2}(ab + ba) = g^{-1}(a, b). \tag{114}$$

We also define the *exterior product* ($\forall r, s = 0, 1, 2, 3$) by

$$\begin{aligned} A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s}, \\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r \end{aligned} \tag{115}$$

where $\langle\rangle_k$ is the component in the subspace $\bigwedge^k T^*M$ of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{C}\ell(T^*M)$.

[30] If the reader need more detail on the Clifford calculus of multivetors he may consult, e.g., [21, 22, 23, 24, 40, 41, 42].

For $A_r = a_1 \wedge ... \wedge a_r, B_r = b_1 \wedge ... \wedge b_r$, the *scalar product* is defined as

$$
\begin{aligned}
A_r \cdot B_r &= (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r) \\
&= \det \begin{bmatrix} a_1 \cdot b_1 & \ldots & a_1 \cdot b_k \\ \ldots & \ldots & \ldots \\ a_k \cdot b_1 & \ldots & a_k \cdot b_k \end{bmatrix}.
\end{aligned} \tag{116}
$$

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r \neq s$, $A_r \cdot B_s = 0$.

For $r \leq s, A_r = a_1 \wedge ... \wedge a_r, B_s = b_1 \wedge ... \wedge b_s$ we define the *left contraction* by

$$
\lrcorner : (A_r, B_s) \mapsto A_r \lrcorner B_s = \sum_{i_1 < ... < i_r} \epsilon^{i_1 ..... i_s}_{1......s} (a_1 \wedge ... \wedge a_r) \cdot (b_{i_1} \wedge ... \wedge b_{i_r})^{\sim} b_{i_r+1} \wedge ... \wedge b_{i_s}, \tag{117}
$$

where $\sim$ denotes the reverse mapping (*reversion*)

$$
\sim : \sec \bigwedge\nolimits^p T^*M \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1, \tag{118}
$$

and extended by linearity to all sections of $\mathcal{C}\ell(T^*M)$. We agree that for $\alpha, \beta \in \sec \bigwedge^0 T^*M$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0 T^*M,\ A_r, \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M$ then $(\alpha A_r) \lrcorner B_s = A_r \lrcorner (\alpha B_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\mathcal{C}\ell(T^*M)$, i.e., for $A, B \in \sec \mathcal{C}\ell(T^*M)$

$$
A \lrcorner B = \sum_{r,s} \langle A \rangle_r \lrcorner \langle B \rangle_s,\ r \leq s. \tag{119}
$$

It is also necessary to introduce in $\mathcal{C}\ell(T^*M)$ the operator of *right contraction* denoted by $\llcorner$. The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $A_r, \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M$ then $A_r \llcorner (\alpha B_s) = (\alpha A_r) \llcorner B_s$.

A.2 Some useful formulas

The main formulas used in the Clifford calculus in the main text can be obtained from the following ones, where $a \in \sec \bigwedge^1 T^*M$ and $A_r, \in \sec \bigwedge^r T^*M, B_s \in$

$\sec \bigwedge^s T^*M$:

$$\begin{aligned} aB_s &= a \lrcorner B_s + a \wedge B_s, B_s a = B_s \llcorner a + B_s \wedge a, \\ a \lrcorner B_s &= \frac{1}{2}(aB_s - (-)^s B_s a), \\ A_r \lrcorner B_s &= (-)^{r(s-1)} B_s \llcorner A_r, \\ a \wedge B_s &= \frac{1}{2}(aB_s + (-)^s B_s a), \\ A_r B_s &= \langle A_r B_s \rangle_{|r-s|} + \langle A_r \lrcorner B_s \rangle_{|r-s-2|} + ... + \langle A_r B_s \rangle_{|r+s|} \\ &= \sum_{k=0}^{m} \langle A_r B_s \rangle_{|r-s|+2k}, \ m = \frac{1}{2}(r+s-|r-s|), \\ A_r \lrcorner B_r &= A_r \llcorner B_r = \tilde{A}_r \cdot B_r = A_r \cdot \tilde{B}_r \end{aligned} \tag{120}$$

A.3 Hodge star operator

Let $\star$ be the usual Hodge star operator $\star : \bigwedge^k T^*M \to \bigwedge^{4-k} T^*M$. If $B \in \sec \bigwedge^k T^*M$, $A \in \sec \bigwedge^{4-k} T^*M$ and $\tau \in \sec \bigwedge^4 T^*M$ is the volume form, then $\star B$ is defined by

$$A \wedge \star B = (A \cdot B)\tau.$$

Then we can show that if $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(T*M)$ we have

$$\star A_p = \widetilde{A_p}\theta^{\mathbf{5}}. \tag{121}$$

This equation is enough to prove very easily the following identities (which are used in the main text):

$$\begin{aligned} A_r \wedge \star B_s &= B_s \wedge \star A_r; \quad r = s, \\ A_r \lrcorner \star B_s &= B_s \lrcorner \star A_r; \quad r + s = 4, \\ A_r \wedge \star B_s &= (-1)^{r(s-1)} \star (\tilde{A}_r \lrcorner B_s); \quad r \leq s, \\ A_r \lrcorner \star B_s &= (-1)^{rs} \star (\tilde{A}_r \wedge B_s); \quad r + s \leq 4 \end{aligned} \tag{122}$$

Let d and δ be respectively the differential and Hodge codifferential operators acting on sections of $\bigwedge T^*M$. If $\omega_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$, then $\delta\omega_p = (-)^p \star^{-1} d \star \omega_p$, where $\star^{-1}\star =$ identity. When applied to a p-form we have

$$\star^{-1} = (-1)^{p(4-p)+1} \star \quad .$$

A.4 Action of $D_{\mathbf{e_a}}$ on Sections of $\mathcal{C}\ell(TM)$ and $\mathcal{C}\ell(T^*M)$

Let $D_{\mathbf{e_a}}$ be a metrical compatible covariant derivative operator acting on sections of the tensor bundle. It can be easily shown (see, e.g., [11]) that $D_{\mathbf{e_a}}$ is also a covariant derivative operator on the Clifford bundles $\mathcal{C}\ell(TM)$ and $\mathcal{C}\ell(T^*M)$.

Now, if $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M)$ we can show, very easily by explicitly performing the calculations[31] that

$$D_{\mathbf{e_a}} A_p = \partial_{\mathbf{e_a}} A_p + \frac{1}{2}[\omega_{\mathbf{e_a}}, A_p], \tag{123}$$

where the $\omega_{\mathbf{e_a}} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M)$ may be called *Clifford connection 2-forms.* They are given by:

$$\omega_{\mathbf{e_a}} = \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}} \theta_{\mathbf{b}} \theta_{\mathbf{c}} = \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}} \theta_{\mathbf{b}} \wedge \theta_{\mathbf{c}}, \tag{124}$$

where (in standard notation)

$$D_{\mathbf{e_a}} \theta_{\mathbf{b}} = \omega_{\mathbf{ab}}^{\mathbf{c}} \theta_{\mathbf{c}}, \quad D_{\mathbf{e_a}} \theta^{\mathbf{b}} = -\omega_{\mathbf{ac}}^{\mathbf{b}} \theta^{\mathbf{c}}, \;\; \omega_{\mathbf{a}}^{\mathbf{bc}} = -\omega_{\mathbf{a}}^{\mathbf{cb}} \tag{125}$$

An analogous formula to Eq.(123) is valid for the covariant derivative of sections of $\mathcal{C}\ell(TM)$ and they were used in several places in the main text.

A.5 Dirac Operator, Differential and Codifferential

The Dirac operator acting on sections of $\mathcal{C}\ell(T^*M)$ is the invariant first order differential operator

$$\boldsymbol{\partial} = \theta^{\mathbf{a}} D_{\mathbf{e_a}}, \tag{126}$$

and we can show (see, e.g., [47]) that when $D_{\mathbf{e_a}}$ is the Levi-Civita covariant derivative operator, the following important result holds:

$$\boldsymbol{\partial} = \boldsymbol{\partial} \wedge + \boldsymbol{\partial} \lrcorner = d - \delta. \tag{127}$$

The square of the Dirac operator $\boldsymbol{\partial}^2$ is called the *Hodge Laplacian.* We have

$$\boldsymbol{\partial}^2 = -(d\delta + \delta d). \tag{128}$$

This operator is not to be confused with the *covariant D'Alembertian* which is given by

$$\Box = \boldsymbol{\partial} \cdot \boldsymbol{\partial}. \tag{129}$$

The following identities were used in the text

$$\begin{aligned}
dd &= \delta\delta = 0, \\
d\boldsymbol{\partial}^2 &= \boldsymbol{\partial}^2 d; \quad \delta\boldsymbol{\partial}^2 = \boldsymbol{\partial}^2 \delta, \\
\delta\star &= (-1)^{p+1} \star d; \quad \star\delta = (-1)^p \star d, \\
d\delta\star &= \star d\delta; \quad \star d\delta = \delta d\star; \quad \star\boldsymbol{\partial}^2 = \boldsymbol{\partial}^2\star
\end{aligned} \tag{130}$$

[31] A derivation of this formula from the genral theory of connections can be found in [39].

A.6 Maxwell Equation

Maxwell equations in the Clifford bundle of differential forms resume in one single equation. Indeed, if $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$ is the electromagnetic field and $J_e \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$ is the electromagnetic current, we have Maxwell equation[32]:

$$\boldsymbol{\partial} F = J_e. \tag{131}$$

Eq.(131) is equivalent to the pair of equations

$$dF = 0, \tag{132}$$
$$\delta F = -J_e. \tag{133}$$

Eq.(132) is called the homogenous equation and Eq.(133) is called the non-homogeneous equation. Note that it can be written also as:

$$d \star F = - \star J_e. \tag{134}$$

B Einstein Field Equations for the Tetrad Fields $\theta^{\mathbf{a}}$

In the main text we gave a Clifford bundle formulation of the field equations of general relativity in a form that resembles a $Sl(2,\mathbb{C})$ gauge theory and also a formulation in terms of a set of 2-form fields $\star\mathcal{S}^{\mathbf{a}}$. Here we want to *recall* yet another face of Einstein's equations, i.e., we show how to write the field equations directly for the tetrad fields $\theta^{\mathbf{a}}$ in such a way that the obtained equations are equivalent to Einstein's field equations. This is done in order to compare the correct equations for that objects which some other equations proposed for these objects that appeared recently in the literature (and which will be discussed below). Before proceeding, we mention that, of course, we could write analogous (and equivalent) equations for the dual tetrads $\mathbf{e}_{\mathbf{a}}$.

As shown in details in papers [47, 58] the correct wave like equations satisfied by the $\theta^{\mathbf{a}}$ are[33]:

$$-(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\theta^{\mathbf{a}} + \boldsymbol{\partial}\wedge(\boldsymbol{\partial}\cdot\theta^{\mathbf{a}}) + \boldsymbol{\partial}\lrcorner(\boldsymbol{\partial}\wedge\theta^{\mathbf{a}}) = \mathcal{T}^{\mathbf{a}} - \frac{1}{2}T\theta^{\mathbf{a}}. \tag{135}$$

In Eq.(135), $\mathcal{T}^{\mathbf{a}} = T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$ are the energy momentum 1-form fields and $T = T^{\mathbf{a}}_{\mathbf{a}} = -R = -R^{\mathbf{a}}_{\mathbf{a}}$, with $T_{\mathbf{ab}}$ the energy momentum tensor of matter. When $\theta^{\mathbf{a}}$ is an exact differential, and in this case

[32] Then, there is no misprint in the title of this section.

[33] Of course, there are analogous equations for the $\mathbf{e}_{\mathbf{a}}$ [30], where in that case, the Dirac operator must be defined (in an obvious way) as acting on sections of the Clifford bundle of multivectors, that has been introduced in section 3.

we write $\theta^{\mathbf{a}} \mapsto \theta^{\mu} = dx^{\mu}$ and if the coordinate functions are harmonic, i.e., $\delta\theta^{\mu} = -\boldsymbol{\partial}\theta^{\mu} = 0$, Eq.(135) becomes

$$\square\theta^{\mu} + \frac{1}{2}R\theta^{\mu} = -\mathcal{T}^{\mu}, \tag{136}$$

where we have used Eq.(129).

In Eq.(135) $\boldsymbol{\partial} = \theta^{\mathbf{a}} D_{\mathbf{e}_a} = \boldsymbol{\partial}\wedge + \boldsymbol{\partial}\lrcorner = d - \delta$ is the Dirac (like) operator acting on sections of the Clifford bundle $\mathcal{C}\ell(T^*M)$ defined in the previous Appendix.

With these formulas we can write

$$\begin{aligned} \boldsymbol{\partial}^2 &= \boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial}, \\ \boldsymbol{\partial}\wedge\boldsymbol{\partial} &= -\boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge, \end{aligned} \tag{137}$$

with

$$\begin{aligned} \boldsymbol{\partial}\cdot\boldsymbol{\partial} &= \eta^{\mathbf{ab}}(D_{\mathbf{e_a}} D_{\mathbf{e_b}} - \omega^{\mathbf{c}}_{\mathbf{ab}} D_{\mathbf{e_c}}), \\ \boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}(D_{\mathbf{e_a}} D_{\mathbf{e_b}} - \omega^{\mathbf{c}}_{\mathbf{ab}} D_{\mathbf{e_c}}). \end{aligned} \tag{138}$$

Note that $D_{e_a}\theta^{\mathbf{b}} = -\omega^{\mathbf{b}}_{\mathbf{ac}}\theta^{\mathbf{c}}$ and a somewhat long, but simple calculation [34] shows that

$$(\boldsymbol{\partial}\wedge\boldsymbol{\partial})\theta^{\mathbf{a}} = \mathcal{R}^{\mathbf{a}}, \tag{139}$$

where, as already defined, $\mathcal{R}^{\mathbf{a}} = R^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}$ are the Ricci 1-forms. We also observe (that for the best of our knowledge) $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ that has been named the Ricci operator in [58] has no analogue in classical differential geometry.

Note that Eq.(135) can be written after some algebra as

$$\mathcal{R}^{\mu} - \frac{1}{2}R\theta^{\mu} = \mathcal{T}^{\mu}, \tag{140}$$

with $\mathcal{R}^{\mu} = R^{\mu}_{\nu}dx^{\nu}$ and $\mathcal{T}^{\mu} = T^{\mu}_{\nu}dx^{\nu}$, $\theta^{\mu} = dx^{\mu}$ in a coordinate chart of the maximal atlas of M covering an open set $U \subset M$.

We are now prepared to make some crucial comments concerning some recent papers ([10],[16]-[19]).

(**i**) In ([10],[16]-[19]) authors claims that the $\mathbf{e_a}$, $\mathbf{a} = 0,1,2,3$ satisfy the equations

$$\boxed{(\square + T)\mathbf{e_a} = 0.}$$

They thought to have produced a valid derivation for that equations. We will not comment on that derivation here. Enough is to say that if that equation was true it would imply that $(\square + T)\theta^{\mathbf{a}} = 0$. This is not the case. Indeed, as a careful reader may verify, the true equation satisfied by any one of the $\theta^{\mathbf{a}}$ is Eq.(135).

[34] The calculation is done in detail in [47, 58].

(**ii**) We quote that author of [16, 17, 18] explicitly wrote several times that the "electromagnetic potential"[35] $\mathbf{A}$ in their theory (a 1-form with values in a vector space) satisfies the following wave equation,

$$\boxed{(\Box + T)\mathbf{A} = 0.}$$

Now, this equation cannot be correct even for the usual $U(1)$ gauge potential of classical electrodynamics [36] $A \in \sec \bigwedge^1 T^*M \subset \sec \mathcal{C}\ell(T^*M)$. Indeed, in vacuum Maxwell equation reads (see Eq.(131))

$$\boldsymbol{\partial} F = 0, \tag{141}$$

where $F = \boldsymbol{\partial} A = \boldsymbol{\partial} \wedge A = dA$, *if* we work in the Lorenz gauge $\boldsymbol{\partial} \cdot A = \boldsymbol{\partial} \lrcorner A = -\delta A = 0$. Now, since we have according to Eq.(128) that $\boldsymbol{\partial}^2 = -(d\delta + \delta d)$,we get

$$\boldsymbol{\partial}^2 A = 0. \tag{142}$$

A simple calculation then shows that in the coordinate basis introduced above we have,

$$(\boldsymbol{\partial}^2 A)_\alpha = g^{\mu\nu} D_\mu D_\nu A_\alpha + R^\nu_\alpha A_\nu \tag{143}$$

and we see that Eq.(142) reads in components[37]

$$\Box A_\mu + R^\nu_\mu A_\nu = 0. \tag{144}$$

Eq.(144) can be found, e.g., in Eddington's book [15] on page 175.

Finally we make a single comment on reference [10], because this paper is related to Sachs 'unified' theory in the sense that authors try to identify Sachs 'electromagnetic' field (discussed in the main text) with a supposedly existing longitudinal electromagnetic field predict by their theory. Well, on [10] we can read at the beginning of section 1.1:

"The antisymmetrized form of special relativity [1] has spacetime metric given by the enlarged structure

$$\eta^{\mu\nu} = \frac{1}{2}\left(\sigma^\mu \sigma^{\nu *} + \sigma^\nu \sigma^{\mu *}\right), \tag{1.1.}$$

where σ^μ are the Pauli matrices satisfying a clifford (sic) algebra

$$\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu},$$

which are represented by

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

[35] In [16, 17, 18] author do identify their "electromagnetic potential" with the bivector valued connection 1-form $\boldsymbol{\omega}$ that we introduced in section above. As we explained with details this cannot be done because that quantity is related to gravitation, not electromagnetism.

[36] Which must be one of the gauge components of the gauge field.

[37] Take into account that in Einstein theory the term $R^\nu_\mu A_\nu = 0$ in vacuum.

The $*$ operator denotes quaternion conjugation, which translates to a spatial parity transformation."

Well, we comment as follows: the $*$ is not really defined anywhere in [10]. If it refers to a spatial parity operation, we infer that $\sigma^{0*} = \sigma^0$ and $\sigma^{i*} = -\sigma^i$. Also, $\eta^{\mu\nu}$ is not defined, but Eq.(3.5) of [10] make us to infers that $\eta^{\mu\nu}$ =diag$(1, -1, -1, 1)$. In that case Eq.(1.1) above (with the first member understood as $\eta^{\mu\nu}\sigma^0$) is true but the equation $\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu}$ is false. Enough is to see that $\{\sigma^0, \sigma^i\} = 2\sigma^i \neq 2\delta^{0i}$.

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